

# Cumulative-Separable Codes

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**Abstract**—**q-ary cumulative-separable  $\Gamma(L, G^{(j)})$ -codes**  $L = \{\alpha \in GF(q^m) : G(\alpha) \neq 0\}$  and  $G^{(j)}(x) = G(x)^j, 1 \leq j \leq q$  are considered. The relation between different codes from this class is demonstrated. Improved boundaries of the minimum distance and dimension are obtained.

**Index Terms**—Goppa codes, cumulative codes, separable codes.

## I. INTRODUCTION

At first  $\Gamma(L, G)$ -codes were introduced by V.D.Goppa [1] in 1970. These codes are a large and powerful class of error correcting codes. F.J. McWilliams and N.J. Sloane [2] defined these codes as the most important class of alternant codes. It is known that there are  $\Gamma(L, G)$ -codes that reach the Gilbert-Varshamov bound and that many  $\Gamma(L, G)$ -codes are placed in the Table of the best known codes [3]. It is noted also that Goppa codes are interesting for postquantum cryptography. There are four basic types of  $\Gamma(L, G)$ -codes: cyclic, separable, cumulative, and irreducible Goppa codes. In this paper we describe new type of  $\Gamma(L, G)$ -codes that we call cumulative-separable Goppa codes. We are motivated to study this class of Goppa codes, because, as it will be shown below, there are its subclasses that have improved estimations on minimum distance and dimension and that there exist codes of these subclasses that have parameters better than those for codes from the Table of the best known codes [3].

This paper is organized as follows. In Section II we review briefly the definitions that we will use in the paper. In Section III we describe subclasses of cumulative-separable  $\Gamma(L, G)$ -codes with improved estimations on the dimension and minimum distance. In Section IV the relations between codes from different subclasses of cumulative-separable codes are presented. In Sections V and VI theorems on estimations of the dimension and minimum distance of considered subclasses are presented.

## II. CLASS OF CUMULATIVE-SEPARABLE $\Gamma(L, G)$ -CODES

### A. Cumulative Goppa codes

**Definition 1:** [1] A Goppa code with  $G(x) = (x - \alpha)^t$ , where  $\alpha \in GF(q^m)$  is called a cumulative code.

It is well known [1], [2] that the cumulative code  $\Gamma(\{GF(q^m) \setminus \{\alpha\}\}, (x - \alpha)^t)$  is equivalent to a cumulative  $\Gamma(\{GF(q^m) \setminus \{0\}\}, x^t)$ , that, in its turn, is equivalent to a primitive BCH-code of length  $n = q^m - 1$  with a parity check matrix

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$$H = \begin{bmatrix} 1 & \alpha^{-t} & \alpha^{-2t} & \dots & \alpha^{-(n-1)t} \\ 1 & \alpha^{-(t-1)} & \alpha^{-2(t-1)} & \dots & \alpha^{-(n-1)(t-1)} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \alpha^{-1} & \alpha^{-2} & \dots & \alpha^{-(n-1)} \end{bmatrix}.$$

**Lemma 1:** A cumulative  $q$ -ary Goppa  $\Gamma(L, G)$ -code with  $L \subset GF(q^m)$  and  $G(x) = x^q$  is equivalent to the  $\Gamma(L, G^*)$ -code with  $G^*(x) = x^{q-1}$ .

*Proof:*

Let us consider a parity check matrix  $H^*$  of the  $\Gamma(L, G^*)$ -code:

$$H^* = \begin{bmatrix} \frac{1}{\alpha_1^{q-1}} & \frac{1}{\alpha_2^{q-1}} & \dots & \frac{1}{\alpha_n^{q-1}} \\ \frac{\alpha_1}{\alpha_1^{q-1}} & \frac{\alpha_2}{\alpha_2^{q-1}} & \dots & \frac{\alpha_n}{\alpha_n^{q-1}} \\ \dots & \dots & \dots & \dots \\ \frac{\alpha_1^{q-2}}{\alpha_1^{q-1}} & \frac{\alpha_2^{q-2}}{\alpha_2^{q-1}} & \dots & \frac{\alpha_n^{q-2}}{\alpha_n^{q-1}} \end{bmatrix}.$$

It is clear that the row

$$h_0 = \left[ \frac{1}{\alpha_1^q} \quad \frac{1}{\alpha_2^q} \quad \dots \quad \frac{1}{\alpha_n^q} \right] = \left[ \frac{1}{\alpha_1} \quad \frac{1}{\alpha_2} \quad \dots \quad \frac{1}{\alpha_n} \right]^q$$

that was obtained by raising to the  $q$ -th power of each component in the last row of the parity check matrix  $H^*$  is also a parity check row for the  $\Gamma(L, G^*)$ -code. Thus, the matrix  $H^*$  can be rewritten as follows:

$$H^* = \begin{bmatrix} \frac{1}{\alpha_1^q} & \frac{1}{\alpha_2^q} & \dots & \frac{1}{\alpha_n^q} \\ \frac{\alpha_1}{\alpha_1^q} & \frac{\alpha_2}{\alpha_2^q} & \dots & \frac{\alpha_n}{\alpha_n^q} \\ \dots & \dots & \dots & \dots \\ \frac{\alpha_1^{q-1}}{\alpha_1^q} & \frac{\alpha_2^{q-1}}{\alpha_2^q} & \dots & \frac{\alpha_n^{q-1}}{\alpha_n^q} \end{bmatrix}.$$

It is easy to see that the matrix  $H^*$  is also a parity check matrix  $H$  for the  $\Gamma(L, G^*)$ -code. ■

**Corollary 1:** The cumulative  $q$ -ary  $\Gamma(L, G)$ -code with  $L \subset GF(q^m)$  and  $G(x) = (x - \alpha)^q, \alpha \in GF(q^m)$  is equivalent to the  $\Gamma(L, G^*)$ -code with  $G^*(x) = (x - \alpha)^{q-1}$ .

### B. Cumulative-separable Goppa codes

**Definition 2:** Goppa code with  $G^{(j)}(x) = G(x)^j$  and  $L \subseteq \{GF(q^m) \setminus \{\alpha : G(\alpha) = 0\}\}$ , where  $G(x)$  is a separable polynomial over  $GF(q^m)$  will be called a cumulative-separable Goppa code. Parameter  $j$  is called a cumulativity order of a cumulative-separable code.

**Corollary 2:** The cumulative-separable  $\Gamma(L, G^{(j)}(x))$ -code can be represented as a common subcode of the cumulative  $\Gamma(L, G_i^{(j)})$ -codes, where

$$G_i^{(j)}(x) = (x - \beta_i)^j \text{ and } G(x) = \prod_{i=1}^{\tau} (x - \beta_i), \\ \tau = \deg G(x), \beta_i \in GF(q^{mt}).$$

**Lemma 2:** The dimension of the cumulative-separable  $\Gamma(L, G^{(j)})$ -code with  $L \subseteq GF(q^m)$  and  $G^{(j)}(x) =$

$\prod_{i=1}^l G_i^{(j)}(x)$  is determined by the dimension of the codes  $\Gamma_1(L, G_1^{(j)}), \Gamma_2(L, G_2^{(j)}), \dots, \Gamma_l(L, G_l^{(j)})$  as follows:

$$k \geq n - \sum_{i=1}^l r_i,$$

where  $r_i$  is the redundancy of the  $\Gamma_i(L, G_i^{(j)})$ -code.

*Proof:*

It is clear that the parity check matrix of  $\Gamma(L, G^{(j)})$ -code can be defined as:

$$H = \begin{bmatrix} H_1 \\ H_2 \\ \dots \\ H_l \end{bmatrix},$$

where  $H_i, i = 1, \dots, l$  are parity check matrices of  $\Gamma_i(L, G_i^{(j)})$  codes. ■

**Lemma 3:** The dimension of  $q$ -ary  $\Gamma(L, G^{(q)})$ -code of length  $n \leq q^m$  with  $L \subseteq GF(q^m)$  and  $G^{(q)}(x) = (G(x))^q$ , where  $\deg G(x) = r$  and  $G(x)$  is a separable polynomial, is determined by the inequality:

$$k \geq n - m(q-1)r.$$

*Proof:*

It follows from Lemma 1 that the  $\Gamma(L, G^{(q)})$ -code is equivalent to the  $\Gamma(L, G^{(q-1)})$ -code with  $G^{(q-1)}(x) = (G(x))^{q-1}$  whose dimension is determined by the following inequality according to Lemma 2:

$$k \geq n - m(q-1)r. \quad \blacksquare$$

### III. SUBCLASSES OF CUMULATIVE-SEPARABLE CODES WITH IMPROVED ESTIMATIONS ON THE MINIMUM DISTANCE AND DIMENSION

Let us consider a family of embedded cumulative-separable  $\Gamma(L, G^{(j)})$  codes with  $L \subseteq GF(q^{2l})$  and  $G^{(j)}(x) = (G(x))^j$ , where  $j = 2, \dots, q-1$  and the degree of the polynomial  $G(x)$  is equal to  $\tau$ . The parity check matrix  $H_j$  for  $\Gamma(L, G^{(j)})$ -code is written as:

$$H_j = \begin{bmatrix} \frac{1}{G(\alpha_1)^j} & \frac{1}{G(\alpha_2)^j} & \dots & \frac{1}{G(\alpha_n)^j} \\ \frac{\alpha_1^{\tau-1}}{G(\alpha_1)^j} & \frac{\alpha_2^{\tau-1}}{G(\alpha_2)^j} & \dots & \frac{\alpha_n^{\tau-1}}{G(\alpha_n)^j} \\ \dots & \dots & \dots & \dots \\ \frac{\alpha_1^{j\tau-1}}{G(\alpha_1)^j} & \frac{\alpha_2^{j\tau-1}}{G(\alpha_2)^j} & \dots & \frac{\alpha_n^{j\tau-1}}{G(\alpha_n)^j} \\ \dots & \dots & \dots & \dots \\ \frac{\alpha_1^{(j-1)\tau-1}}{G(\alpha_1)^j} & \frac{\alpha_2^{(j-1)\tau-1}}{G(\alpha_2)^j} & \dots & \frac{\alpha_n^{(j-1)\tau-1}}{G(\alpha_n)^j} \\ \dots & \dots & \dots & \dots \\ \frac{\alpha_1^{j\tau-1}}{G(\alpha_1)^j} & \frac{\alpha_2^{j\tau-1}}{G(\alpha_2)^j} & \dots & \frac{\alpha_n^{j\tau-1}}{G(\alpha_n)^j} \end{bmatrix}. \quad (1)$$

Using the linear combination of corresponding rows of the matrix  $H_j$  we can present it in the following form:

$$H_j = \begin{bmatrix} \frac{1}{G(\alpha_1)^j} & \frac{1}{G(\alpha_2)^j} & \dots & \frac{1}{G(\alpha_n)^j} \\ \frac{\alpha_1^{\tau-1}}{G(\alpha_1)^j} & \frac{\alpha_2^{\tau-1}}{G(\alpha_2)^j} & \dots & \frac{\alpha_n^{\tau-1}}{G(\alpha_n)^j} \\ \dots & \dots & \dots & \dots \\ \frac{\alpha_1^{j\tau-1}}{G(\alpha_1)^j} & \frac{\alpha_2^{j\tau-1}}{G(\alpha_2)^j} & \dots & \frac{\alpha_n^{j\tau-1}}{G(\alpha_n)^j} \\ \dots & \dots & \dots & \dots \\ \frac{\alpha_1^{(j-1)\tau-1}}{G(\alpha_1)^j} & \frac{\alpha_2^{(j-1)\tau-1}}{G(\alpha_2)^j} & \dots & \frac{\alpha_n^{(j-1)\tau-1}}{G(\alpha_n)^j} \end{bmatrix}. \quad (2)$$

Thus, the parity check matrix of the  $\Gamma(L, G^{(j+1)})$ -code is obtained from that of the preceding  $\Gamma(L, G^{(j)})$ -code by writing up a submatrix  $h_{j+1}$ :

$$h_{j+1} = \begin{bmatrix} \frac{1}{G(\alpha_1)^{j+1}} & \frac{1}{G(\alpha_2)^{j+1}} & \dots & \frac{1}{G(\alpha_n)^{j+1}} \\ \frac{\alpha_1^{\tau-1}}{G(\alpha_1)^{j+1}} & \frac{\alpha_2^{\tau-1}}{G(\alpha_2)^{j+1}} & \dots & \frac{\alpha_n^{\tau-1}}{G(\alpha_n)^{j+1}} \\ \dots & \dots & \dots & \dots \\ \frac{\alpha_1^{j\tau-1}}{G(\alpha_1)^{j+1}} & \frac{\alpha_2^{j\tau-1}}{G(\alpha_2)^{j+1}} & \dots & \frac{\alpha_n^{j\tau-1}}{G(\alpha_n)^{j+1}} \end{bmatrix}.$$

Respectively, the parity check matrix  $H_j$  can be rewritten as follows:

$$H_j = \begin{bmatrix} h_1 \\ h_2 \\ \dots \\ h_j \end{bmatrix}.$$

We can obtain the following corollary from the above relations.

**Corollary 3:** The cumulative-separable  $\Gamma(L, G^{(j)})$ -code is a subcode of the cumulative-separable  $\Gamma(L, G^{(j-1)})$ -code.

Consider now cumulative-separable codes with the improved estimation on dimension that are obtained from the following separable Goppa codes [4], [5]:

$$\begin{aligned} \Gamma_1 &= \Gamma(L_1, G_1), \quad L_1 = \{GF(t^2) \setminus \{\alpha : G_1(\alpha) = 0\}\} \\ &\quad \text{and } G_1(x) = x^{t-1} + 1; \\ \Gamma_1^* &= \Gamma(L_1^*, G_1), \quad L_1^* = \{L_1 \setminus \{0\}\} \\ &\quad \text{and } G_1(x) = x^{t-1} + 1; \\ \Gamma_2 &= \Gamma(L_2, G_2), \quad L_2 = L_1^* \\ &\quad \text{and } G_2(x) = x^t + x; \\ \Gamma_3 &= \Gamma(L_3, G_3), \quad L_3 = \{GF(t^2) \setminus \{\alpha : G_3(\alpha) = 0\}\} \\ &\quad \text{and } G_3(x) = x^t + x + 1; \\ \Gamma_4^* &= \Gamma(L_4^*, G_4), \quad L_4^* = GF(t^2) \setminus \{\alpha : G_4(\alpha) = 0\} \cup \{0\} \\ &\quad \text{and } G_4(x) = x^t + x^{t-1} + 1; \\ \Gamma_5 &= \Gamma(L_5, G_5), \quad L_5 = L_4^* \\ &\quad \text{and } G_5(x) = x^{t+1} + x^t + x; \\ \Gamma_6 &= \Gamma(L_6, G_6), \quad L_6 = \{GF(t^2) \setminus \{\alpha : G_6(\alpha) = 0\}\} \\ &\quad \text{and } G_6(x) = x^{t+1} + 1. \end{aligned}$$

As it has been shown earlier in [4], [5], these codes are related by following relations that allow us to speak about a chain of equivalent and embedded codes.

- The  $\Gamma_1^*$ -code is obtained from the  $\Gamma_1$ -code by shortening in information symbol corresponding to the numerator

$\{0\}$  from the set  $L_1$ . The parameters of these codes are related by following relations:  $n_1^* = n_1 - 1 = (t^2 - t + 1) - 1$ ,  $r_1^* = r_1$ ,  $k_1^* = k_1 - 1$ ,  $d_1^* \geq d_1 = t$ .

- The code  $\Gamma_2$  is a subcode of  $\Gamma_1^*$ .  $L_2 = L_1^* = L_1 \setminus \{0\}$ , i.e.,  $n_2 = n_1^* = n_1 - 1$  and  $G_2(x) = xG_1(x)$ . The parity check matrices of these codes are related by the following relation:

$$H_2 = \begin{bmatrix} \frac{1}{G_2(\alpha_{i_1})} \cdots \frac{1}{G_2(\alpha_{i_{n_2}})} \\ H_1^* \end{bmatrix},$$

where  $H_1^*$  is the parity check matrix of the code  $\Gamma_1^*$ , with the redundancy  $r_1^* = r_1$ . Hence, the redundancy of the code  $\Gamma_2$  is determined by the inequality  $r_2 \leq r_1 + l$ . The minimum distance is  $d_2 \geq \deg G_2(x) + 1 = t + 1$ .

- The code  $\Gamma_3$  is equivalent to code  $\Gamma_2$  up to the permutation

$$\gamma \rightarrow \gamma + \beta, \text{ where the element } \beta \in GF(t^2) \text{ is such that } \beta^t + \beta = 1 \text{ and } \gamma \in L_2.$$

It follows that  $n_3 = n_2$ ,  $k_3 = k_2$ ,  $d_3 = d_2$ .

- An auxiliary code  $C_3^*$  is obtained by shortening of the code  $\Gamma_3$  in a redundancy symbol corresponding to the numerator  $\{0\} \in L_3$ . Thus, the parity check matrix  $H_3^*$  of the code  $C_3^*$  results from deleting the first unit row and the last column of the parity check matrix  $H_3$  of the code  $\Gamma_3$ :

$$H_3 = \begin{bmatrix} 1 & \cdots & 1 & 1 \\ & H_3^* & & 0 \end{bmatrix}.$$

Therefore,  $L_3^* = L_3 \setminus \{0\}$ ,  $n_3^* = n_3 - 1$ ,  $k_3^* = k_3$ ,  $d_3^* \leq d_3$ .

- The auxiliary code  $C_3^*$  is equivalent to the code  $\Gamma_4^*$  up to permutation  $\beta \rightarrow \frac{1}{\beta}$ , where element  $\beta \in L_3^*$  and, hence,  $n_4^* = n_3^*$ ,  $k_4^* = k_3^*$ ,  $d_4^* = d_3^*$ .
- The code  $\Gamma_5$  is a subcode of the code  $\Gamma_4^*$  and their parity check matrices are related as follows:

$$H_5 = \begin{bmatrix} \frac{1}{G_5(\alpha_{i_1})} & \cdots & \frac{1}{G_5(\alpha_{i_{n_5}})} \\ & H_4^* & \end{bmatrix}.$$

The code  $\Gamma_5$  has the following parameters:  $n_5 = n_4^*$ ,  $k_5 = k_4^* - l$ ,  $d_5 \geq d_4^*$ .

- The codes  $\Gamma_5$  and  $\Gamma_6$  are equivalent up to the permutation  $\beta \rightarrow \gamma\beta - 1$ , where  $\gamma \in GF(t^2)$  and  $\gamma^{t+1} + 1 = 0$ ,  $\beta \in L_5$ . Thus,  $n_6 = n_5$ ,  $k_6 = k_5$  and  $d_5 = d_6 = t + 2$ .

Let us transfer now the relations between the above subclasses of separable codes to the case of cumulative-separable codes obtained from these codes.

#### IV. RELATIONS BETWEEN CUMULATIVE-SEPARABLE CODES FROM DIFFERENT SUBCLASSES

A. *Relation between the cumulative-separable codes  $\Gamma_1^{*(i)}$  and  $\Gamma_1^{(i)}$*

- $\Gamma_1^{(i)} = \Gamma(L_1, G_1^{(i)})$  with  $L_1 = \{GF(t^2) \setminus \{\alpha : G_1(\alpha) = 0\}\}$ ,  $n_1^{(i)} = n_1$  and  $G_1^{(i)}(x) = (x^{t-1} + 1)^i$ ;
- $\Gamma_1^{*(i)} = \Gamma(L_1^*, G_1^{(i)})$  with  $L_1^* = \{L_1 \setminus \{0\}\}$ ,  $n_1^{*(i)} = n_1^*$  and  $G_1^{(i)}(x) = (x^{t-1} + 1)^i$ ;

It follows from the above considered definition of these codes that the  $\Gamma_1^{*(i)}$ -code is obtained from the  $\Gamma_1^{(i)}$ -code by shortening in an information symbol corresponding to the numerator  $\{0\}$ .

Parameters of these codes are related as:  $n_1^* = n_1 - 1 = t^2 - t$ ,  $r_1^{*(i)} = r_1^{(i)}$ ,  $k_1^{*(i)} = k_1^{(i)} - 1$ ,  $d_1^{*(i)} \geq d_1^{(i)}$ .

B. *Relation between the cumulative-separable codes  $\Gamma_2^{(i)}$  and  $\Gamma_1^{*(i)}$*

$$\Gamma_2^{(i)} = \Gamma(L_2, G_2^{(i)}) \quad \text{with} \quad L_2 = L_1^*, \\ n_2^{(i)} = n_2 \quad \text{and} \quad G_2^{(i)}(x) = (x^t + x)^i.$$

Two cases should be considered here. They are determined by the cumulativity order  $i$ :

$i < q - 1$ : In this case the  $\Gamma_2^{(i)}$ -code is a subcode of the  $\Gamma_1^{*(i)}$ -code.  $L_2 = L_1^* = L_1 \setminus \{0\}$ , i.e.,  $n_2 = n_1^* = n_1 - 1$  and  $G_2^{(i)}(x) = x^i G_1^{(i)}(x)$ .

*Lemma 4:* The parity check matrices of the codes  $\Gamma_2^{(i)}$  and  $\Gamma_1^{*(i)}$  are related as follows:

$$H_2^{(i)} = \begin{bmatrix} \frac{1}{G_2^{(i)}(\alpha_{i_1})} & \cdots & \frac{1}{G_2^{(i)}(\alpha_{i_{n_2}})} \\ & H_1^{*(i)} & \end{bmatrix},$$

where  $H_1^{*(i)}$  is the parity check matrix of the  $\Gamma_1^{*(i)}$ -code.

*Proof:*

We use representation (1) of the parity check matrix of the cumulative-separable code  $\Gamma_2^{(i)}$ :

$$H_2^{(i)} = \begin{bmatrix} \frac{1}{\alpha_1^i (\alpha_1^{t-1} + 1)^i} & \cdots & \frac{1}{\alpha_{n_2}^i (\alpha_{n_2}^{t-1} + 1)^i} \\ \frac{1}{\alpha_1^{i-1} (\alpha_1^{t-1} + 1)^i} & \cdots & \frac{1}{\alpha_{n_2}^{i-1} (\alpha_{n_2}^{t-1} + 1)^i} \\ \vdots & \cdots & \vdots \\ \frac{1}{\alpha_1 (\alpha_1^{t-1} + 1)^i} & \cdots & \frac{1}{\alpha_{n_2} (\alpha_{n_2}^{t-1} + 1)^i} \\ \frac{1}{(\alpha_1^{t-1} + 1)^i} & \cdots & \frac{1}{(\alpha_{n_2}^{t-1} + 1)^i} \\ \vdots & \cdots & \vdots \\ \frac{\alpha_1^{i(t-1)-1}}{(\alpha_1^{t-1} + 1)^i} & \cdots & \frac{\alpha_{n_2}^{i(t-1)-1}}{(\alpha_{n_2}^{t-1} + 1)^i} \end{bmatrix} \\ = \begin{bmatrix} \frac{1}{\alpha_1^i (\alpha_1^{t-1} + 1)^i} & \cdots & \frac{1}{\alpha_{n_2}^i (\alpha_{n_2}^{t-1} + 1)^i} \\ \frac{1}{\alpha_1^{i-1} (\alpha_1^{t-1} + 1)^i} & \cdots & \frac{1}{\alpha_{n_2}^{i-1} (\alpha_{n_2}^{t-1} + 1)^i} \\ \vdots & \cdots & \vdots \\ \frac{1}{\alpha_1 (\alpha_1^{t-1} + 1)^i} & \cdots & \frac{1}{\alpha_{n_2} (\alpha_{n_2}^{t-1} + 1)^i} \\ & & H_1^{*(i)} \end{bmatrix}. \quad (3)$$

It is easy to see that for component-wise raising to a power  $t$  of any row

$$\left[ \frac{1}{\alpha_1^j (\alpha_1^{t-1} + 1)^i} \quad \cdots \quad \frac{1}{\alpha_{n_2}^j (\alpha_{n_2}^{t-1} + 1)^i} \right], 0 < j < i$$

of the matrix  $H_2^{(i)}$  the corresponding row of the submatrix  $H_1^{*(i)}$  is obtained:

$$\left[ \frac{1}{\alpha_1^j (\alpha_1^{t-1} + 1)^i} \quad \cdots \quad \frac{1}{\alpha_{n_2}^j (\alpha_{n_2}^{t-1} + 1)^i} \right]^t = \left[ \frac{1}{\alpha_1^{jt} (\alpha_1^{t-1} + 1)^i} \quad \cdots \quad \frac{1}{\alpha_{n_2}^{jt} (\alpha_{n_2}^{t-1} + 1)^i} \right] = \left[ \frac{\alpha_1^{t(i-j)-i}}{(\alpha_1^{t-1} + 1)^i} \quad \cdots \quad \frac{\alpha_{n_2}^{t(i-j)-i}}{(\alpha_{n_2}^{t-1} + 1)^i} \right].$$

Thus, the parity check matrix of the  $\Gamma_2^{(i)}$ -code can be written and as

$$H_2^{(i)} = \begin{bmatrix} \frac{1}{\alpha_1^i(\alpha_1^{t-1}+1)^i} & \cdots & \frac{1}{\alpha_{n_2}^i(\alpha_{n_2}^{t-1}+1)^i} \\ & H_1^{*(i)} & \end{bmatrix}, \quad (4)$$

where the first row consists of the elements from  $GF(t)$  only:

$$\begin{aligned} & \left[ \frac{1}{\alpha_1^i(\alpha_1^{t-1}+1)^i} \cdots \frac{1}{\alpha_{n_2}^i(\alpha_{n_2}^{t-1}+1)^i} \right] = \\ & \left[ \frac{1}{(\alpha_1^t+\alpha_1)^t} \cdots \frac{1}{(\alpha_{n_2}^t+\alpha_{n_2})^t} \right] = \\ & \left[ \left( \frac{1}{G_2(\alpha_{i_1})} \right)^i \cdots \left( \frac{1}{G_2(\alpha_{i_{n_2}})} \right)^i \right]. \end{aligned}$$

Hence, the redundancy of the  $\Gamma_2^{(i)}$ -code is determined by the inequality  $r_2^{(i)} \leq r_1^{(i)} + l$ . ■

$i = q-1, i = q$ : By using Corollary 1 and Corollary 2 we obtain that the  $\Gamma_2^{(q-1)}$ -code is equivalent to the  $\Gamma_2^{(q)}$ -code and the  $\Gamma_1^{*(q-1)}$ -code is equivalent to the  $\Gamma_1^{*(q)}$ -code. Let us prove now the equivalency of the codes  $\Gamma_2^{(q-1)}$  and  $\Gamma_1^{*(q-1)}$ .

*Lemma 5:* Parity check matrices of the codes  $\Gamma_2^{(q-1)}$  and  $\Gamma_1^{*(q-1)}$  are equal:

$$H_2^{(q-1)} = H_1^{*(q-1)}.$$

*Proof:*

Using Lemma 4, the relation between the parity check matrices of the codes  $\Gamma_2^{(q-1)}$  and  $\Gamma_1^{*(q-1)}$  can be written as:

$$H_2^{(q-1)} = \begin{bmatrix} \frac{1}{\alpha_1^{q-1}(\alpha_1^{t-1}+1)^{q-1}} & \cdots & \frac{1}{\alpha_{n_2}^{q-1}(\alpha_{n_2}^{t-1}+1)^{q-1}} \\ & H_1^{*(q-1)} & \end{bmatrix}.$$

As the codes  $\Gamma_1^{*(q-1)}$  and  $\Gamma_1^{*(q)}$  are equivalent, then their parity check matrices are equal:  $H_1^{*(q-1)} = H_1^{*(q)}$ . We show that the row

$$\left[ \frac{1}{\alpha_1^{q-1}(\alpha_1^{t-1}+1)^{q-1}} \cdots \frac{1}{\alpha_{n_2}^{q-1}(\alpha_{n_2}^{t-1}+1)^{q-1}} \right]$$

of the matrix  $H_2^{(q-1)}$  can be obtained as the linear combination of the row of the matrix  $H_1^{*(q)}$ :

$$\left[ \frac{\alpha_1^{t-q}}{(\alpha_1^{t-1}+1)^q} \cdots \frac{\alpha_{n_2}^{t-q}}{(\alpha_{n_2}^{t-1}+1)^q} \right]$$

and its  $t$ -th power:

$$\begin{aligned} & \left( \left[ \frac{\alpha_1^{t-q}}{(\alpha_1^{t-1}+1)^q} \cdots \frac{\alpha_{n_2}^{t-q}}{(\alpha_{n_2}^{t-1}+1)^q} \right] \right)^t + \\ & \left[ \frac{\alpha_1^{t-q}}{(\alpha_1^{t-1}+1)^q} \cdots \frac{\alpha_{n_2}^{t-q}}{(\alpha_{n_2}^{t-1}+1)^q} \right] = \\ & \left[ \frac{1}{\alpha_1^{q-1}(\alpha_1^{t-1}+1)^{q-1}} \cdots \frac{1}{\alpha_{n_2}^{q-1}(\alpha_{n_2}^{t-1}+1)^{q-1}} \right]. \end{aligned}$$

Indeed,

$$\begin{aligned} & \left( \left[ \frac{\alpha_1^{t-q}}{(\alpha_1^{t-1}+1)^q} \cdots \frac{\alpha_{n_2}^{t-q}}{(\alpha_{n_2}^{t-1}+1)^q} \right] \right)^t = \\ & \left[ \frac{\alpha_1^{1-qt}}{(\alpha_1^{t-1}+1)^q} \cdots \frac{\alpha_{n_2}^{1-qt}}{(\alpha_{n_2}^{t-1}+1)^q} \right] = \\ & \left[ \frac{1}{\alpha_1^{q-1}(\alpha_1^{t-1}+1)^q} \cdots \frac{1}{\alpha_{n_2}^{q-1}(\alpha_{n_2}^{t-1}+1)^q} \right] \end{aligned}$$

$$\begin{aligned} & \left[ \frac{1}{\alpha_1^{q-1}(\alpha_1^{t-1}+1)^q} \cdots \frac{1}{\alpha_{n_2}^{q-1}(\alpha_{n_2}^{t-1}+1)^q} \right] + \\ & \left[ \frac{\alpha_1^{t-q}}{(\alpha_1^{t-1}+1)^q} \cdots \frac{\alpha_{n_2}^{t-q}}{(\alpha_{n_2}^{t-1}+1)^q} \right] = \\ & \left[ \frac{\alpha_1^{t-1}+1}{\alpha_1^{q-1}(\alpha_1^{t-1}+1)^q} \cdots \frac{\alpha_{n_2}^{t-1}+1}{\alpha_{n_2}^{q-1}(\alpha_{n_2}^{t-1}+1)^q} \right] = \\ & \left[ \frac{1}{\alpha_1^{q-1}(\alpha_1^{t-1}+1)^{q-1}} \cdots \frac{1}{\alpha_{n_2}^{q-1}(\alpha_{n_2}^{t-1}+1)^{q-1}} \right]. \end{aligned}$$

Hence, the redundancy of the  $\Gamma_2^{(q-1)}$ -code coincides with that of the  $\Gamma_1^{*(q-1)}$ -code: ■

$$r_2^{(q-1)} = r_1^{*(q-1)}.$$

1) *Relation between the cumulative-separable codes  $\Gamma_2^{(i)}$  and  $\Gamma_3^{(i)}$ :* Let us define:

$$\Gamma_3^{(i)} = \Gamma(L_3, G_3^{(i)}) \text{ with } L_3 = \{GF(t^3) \setminus \{\alpha : G_3(\alpha) = 0\}\}, \\ n_3^{(i)} = n_3 \text{ and } G_3^{(i)}(x) = (x^t + x + 1)^i.$$

It follows that the codes  $\Gamma_3^{(i)}$  and  $\Gamma_2^{(i)}$  are equivalent for all  $i \geq 1$  with up to permutation

$$\gamma \rightarrow \gamma + \beta, \text{ where element } \beta \in GF(t^2) \\ \text{is such that } \beta^t + \beta = 1 \text{ and } \gamma \in L_2.$$

Hence,  $n_3 = n_2$ ,  $k_3^{(i)} = k_2^{(i)}$ ,  $d_3^{(i)} = d_2^{(i)}$ .

C. *Relation between the cumulative-separable  $\Gamma_3^{(i)}$ -code and auxiliary  $C_3^{*(i)}$ -code*

Let us use representation (1) of the parity check matrix of the  $\Gamma_3^{(i)}$ -code:

$$H_3^{(i)} = \begin{bmatrix} \frac{1}{(\alpha_1^t + \alpha_1 + 1)^i} & \cdots & \frac{1}{(\alpha_{n_3-1}^t + \alpha_{n_3-1} + 1)^i} & 1 \\ \frac{\alpha_1}{(\alpha_1^t + \alpha_1 + 1)^i} & \cdots & \frac{\alpha_{n_3-1}}{(\alpha_{n_3-1}^t + \alpha_{n_3-1} + 1)^i} & 0 \\ \cdots & \cdots & \cdots & 0 \\ \frac{\alpha_1^i}{(\alpha_1^t + \alpha_1 + 1)^i} & \cdots & \frac{\alpha_{n_3-1}^i}{(\alpha_{n_3-1}^t + \alpha_{n_3-1} + 1)^i} & 0 \\ \cdots & \cdots & \cdots & 0 \\ \frac{\alpha_1^{ti-1}}{(\alpha_1^t + \alpha_1 + 1)^i} & \cdots & \frac{\alpha_{n_3-1}^{ti-1}}{(\alpha_{n_3-1}^t + \alpha_{n_3-1} + 1)^i} & 0 \end{bmatrix}, \quad (5)$$

where  $\alpha_{n_3} = 0$ . Therefore,

$$\frac{1}{(\alpha_{n_3}^t + \alpha_{n_3} + 1)^i} = 1.$$

It is clear that component-wise raising to the power  $t$  of the  $(i+1)$ -th row of the parity check matrix gives us the next, i.e., the  $(ti+1)$ -th check row:

$$\begin{aligned} & \left( \left[ \frac{\alpha_1^i}{(\alpha_1^t + \alpha_1 + 1)^i} \cdots \frac{\alpha_{n_3-1}^i}{(\alpha_{n_3-1}^t + \alpha_{n_3-1} + 1)^i} \ 0 \right] \right)^t = \\ & \left[ \frac{\alpha_1^{ti}}{(\alpha_1^t + \alpha_1 + 1)^i} \cdots \frac{\alpha_{n_3-1}^{ti}}{(\alpha_{n_3-1}^t + \alpha_{n_3-1} + 1)^i} \ 0 \right]. \end{aligned}$$

The parity check matrix with the new added row can be rewritten as follows:

$$H_3^{(i)} = \begin{bmatrix} \frac{1}{(\alpha_1^t + \alpha_1 + 1)^i} & \cdots & \frac{1}{(\alpha_{n_3-1}^t + \alpha_{n_3-1} + 1)^i} & 1 \\ \frac{\alpha_1}{(\alpha_1^t + \alpha_1 + 1)^i} & \cdots & \frac{\alpha_{n_3-1}}{(\alpha_{n_3-1}^t + \alpha_{n_3-1} + 1)^i} & 0 \\ \vdots & \vdots & \vdots & 0 \\ \frac{\alpha_1^i}{(\alpha_1^t + \alpha_1 + 1)^i} & \cdots & \frac{\alpha_{n_3-1}^i}{(\alpha_{n_3-1}^t + \alpha_{n_3-1} + 1)^i} & 0 \\ \vdots & \vdots & \vdots & 0 \\ \frac{\alpha_1^{ti-1}}{(\alpha_1^t + \alpha_1 + 1)^i} & \cdots & \frac{\alpha_{n_3-1}^{ti-1}}{(\alpha_{n_3-1}^t + \alpha_{n_3-1} + 1)^i} & 0 \\ \frac{\alpha_1^{ti}}{(\alpha_1^t + \alpha_1 + 1)^i} & \cdots & \frac{\alpha_{n_3-1}^{ti}}{(\alpha_{n_3-1}^t + \alpha_{n_3-1} + 1)^i} & 0 \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 1 & 1 \\ \frac{\alpha_1}{(\alpha_1^t + \alpha_1 + 1)^i} & \cdots & \frac{\alpha_{n_3-1}}{(\alpha_{n_3-1}^t + \alpha_{n_3-1} + 1)^i} & 0 \\ \vdots & \vdots & \vdots & 0 \\ \frac{\alpha_1^i}{(\alpha_1^t + \alpha_1 + 1)^i} & \cdots & \frac{\alpha_{n_3-1}^i}{(\alpha_{n_3-1}^t + \alpha_{n_3-1} + 1)^i} & 0 \\ \vdots & \vdots & \vdots & 0 \\ \frac{\alpha_1^{ti-1}}{(\alpha_1^t + \alpha_1 + 1)^i} & \cdots & \frac{\alpha_{n_3-1}^{ti-1}}{(\alpha_{n_3-1}^t + \alpha_{n_3-1} + 1)^i} & 0 \\ \frac{\alpha_1^{ti}}{(\alpha_1^t + \alpha_1 + 1)^i} & \cdots & \frac{\alpha_{n_3-1}^{ti}}{(\alpha_{n_3-1}^t + \alpha_{n_3-1} + 1)^i} & 0 \end{bmatrix}.$$

Now we can obtain the following relation between the  $H_3^{(i)}$  and  $H_3^{*(i)}$  matrices:

$$H_3^{(i)} = \begin{bmatrix} 1 & \cdots & 1 & 1 \\ H_3^{*(i)} & & & 0 \end{bmatrix}.$$

The auxiliary  $C_3^*$ -code is obtained by shortening of the  $\Gamma_3^{(i)}$ -code in a redundancy symbol corresponding to the numerator  $\{0\} \in L_3$ . Thus, the parity check matrix  $H_3^{*(i)}$  of the  $C_3^*$ -code results from deleting the first unit row and the last column of the parity check matrix  $H_3^{(i)}$  of the  $\Gamma_3^{(i)}$ -code. It follows that  $n_3^{*(i)} = n_3 - 1$ ,  $k_3^{*(i)} = k_3^{(i)}$ ,  $d_3^{*(i)} \leq d_3^{(i)}$ .

#### D. Relation between the auxiliary $C_3^{*(i)}$ -code and cumulative-separable $\Gamma_4^{*(i)}$ -code

We define that

$$\begin{aligned} \Gamma_4^{*(i)} &= \Gamma(L_4^*, G_4^{(i)}) \\ \text{with } L_4^* &= \{GF(t^2) \setminus \{\alpha : G_4(\alpha) = 0\} \cup \{0\}\}, \\ n_4^{*(i)} &= n_4^* \text{ and } G_4^{(i)}(x) = (x^t + x^{t-1} + 1)^i. \end{aligned}$$

The  $C_3^{*(i)}$ -code is equivalent to the  $\Gamma_4^{*(i)}$ -code with up to the permutation  $\alpha_i \rightarrow \frac{1}{\alpha_i}$ ,  $i = 1, \dots, n_3 - 1$ . Indeed, let us write

the  $H_3^{*(i)}$  and perform the following permutation:

$$H_3^{*(i)} = \begin{bmatrix} \frac{\alpha_1}{(\alpha_1^t + \alpha_1 + 1)^i} & \cdots & \frac{\alpha_{n_3-1}}{(\alpha_{n_3-1}^t + \alpha_{n_3-1} + 1)^i} \\ \vdots & \vdots & \vdots \\ \frac{\alpha_1^i}{(\alpha_1^t + \alpha_1 + 1)^i} & \cdots & \frac{\alpha_{n_3-1}^i}{(\alpha_{n_3-1}^t + \alpha_{n_3-1} + 1)^i} \\ \vdots & \vdots & \vdots \\ \frac{\alpha_1^{ti-1}}{(\alpha_1^t + \alpha_1 + 1)^i} & \cdots & \frac{\alpha_{n_3-1}^{ti-1}}{(\alpha_{n_3-1}^t + \alpha_{n_3-1} + 1)^i} \\ \frac{\alpha_1^{ti}}{(\alpha_1^t + \alpha_1 + 1)^i} & \cdots & \frac{\alpha_{n_3-1}^{ti}}{(\alpha_{n_3-1}^t + \alpha_{n_3-1} + 1)^i} \end{bmatrix} \xrightarrow{\alpha_i \rightarrow \frac{1}{\alpha_i}} \begin{bmatrix} \frac{\alpha_1^{ti}}{(\alpha_1^t + \alpha_1^{t-1} + 1)^i} & \cdots & \frac{\alpha_{n_3-1}^{ti}}{(\alpha_{n_3-1}^t + \alpha_{n_3-1}^{t-1} + 1)^i} \\ \vdots & \vdots & \vdots \\ \frac{\alpha_1^{ti-i}}{(\alpha_1^t + \alpha_1^{t-1} + 1)^i} & \cdots & \frac{\alpha_{n_3-1}^{ti-i}}{(\alpha_{n_3-1}^t + \alpha_{n_3-1}^{t-1} + 1)^i} \\ \vdots & \vdots & \vdots \\ \frac{\alpha_1}{(\alpha_1^t + \alpha_1^{t-1} + 1)^i} & \cdots & \frac{\alpha_{n_3-1}}{(\alpha_{n_3-1}^t + \alpha_{n_3-1}^{t-1} + 1)^i} \\ \frac{1}{(\alpha_1^t + \alpha_1^{t-1} + 1)^i} & \cdots & \frac{1}{(\alpha_{n_3-1}^t + \alpha_{n_3-1}^{t-1} + 1)^i} \end{bmatrix}.$$

It is clear that the first row of this matrix is the  $t$ -th power of its last row:

$$\left( \left[ \frac{1}{(\alpha_1^t + \alpha_1^{t-1} + 1)^i} \quad \cdots \quad \frac{1}{(\alpha_{n_3-1}^t + \alpha_{n_3-1}^{t-1} + 1)^i} \right] \right)^t = \left[ \frac{\alpha_1^{ti}}{(\alpha_1^t + \alpha_1^{t-1} + 1)^i} \quad \cdots \quad \frac{\alpha_{n_3-1}^{ti}}{(\alpha_{n_3-1}^t + \alpha_{n_3-1}^{t-1} + 1)^i} \right].$$

It means that the above permutation in the  $C_3^{*(i)}$ -code transforms it into the  $\Gamma_4^{*(i)}$ -code with the parity check matrix  $H_4^{*(i)}$ :

$$H_4^{*(i)} = \begin{bmatrix} \frac{1}{(\alpha_1^t + \alpha_1^{t-1} + 1)^i} & \cdots & \frac{1}{(\alpha_{n_3-1}^t + \alpha_{n_3-1}^{t-1} + 1)^i} \\ \vdots & \vdots & \vdots \\ \frac{\alpha_1^{ti-i}}{(\alpha_1^t + \alpha_1^{t-1} + 1)^i} & \cdots & \frac{\alpha_{n_3-1}^{ti-i}}{(\alpha_{n_3-1}^t + \alpha_{n_3-1}^{t-1} + 1)^i} \\ \vdots & \vdots & \vdots \\ \frac{\alpha_1^{ti-1}}{(\alpha_1^t + \alpha_1^{t-1} + 1)^i} & \cdots & \frac{\alpha_{n_3-1}^{ti-1}}{(\alpha_{n_3-1}^t + \alpha_{n_3-1}^{t-1} + 1)^i} \end{bmatrix}.$$

Thus,  $n_4^{*(i)} = n_4^* = n_3^*$ ,  $k_4^{*(i)} = k_3^{*(i)}$ ,  $d_4^{*(i)} = d_3^{*(i)}$ .

#### E. Relation between the cumulative-separable codes $\Gamma_4^{*(i)}$ and $\Gamma_5^{(i)}$

We define

$$\begin{aligned} \Gamma_5^{(i)} &= \Gamma(L_5, G_5^{(i)}) \text{ with } L_5 = L_4^*, \\ n_5^{(i)} &= n_5 = n_4^* \text{ and } G_5^{(i)}(x) = (x^{t+1} + x^t + x)^i. \end{aligned}$$

Similarly to the case with the  $\Gamma_2^{(i)}$ -code and  $\Gamma_1^{*(i)}$ -code, two cases characterized by the cumulativity order  $i$  should be considered:

**$i < q - 1$  :** In this case the  $\Gamma_5^{(i)}$ -code is a subcode of the  $\Gamma_4^{*(i)}$ -code.

**Lemma 6:** The parity check matrices of the codes  $\Gamma_5^{(i)}$  and  $\Gamma_4^{*(i)}$  are related as:

$$H_5^{(i)} = \begin{bmatrix} \frac{1}{G_5^{(i)}(\alpha_{i1})} & \cdots & \frac{1}{G_5^{(i)}(\alpha_{in_5})} \\ & & H_4^{*(i)} \end{bmatrix},$$

where  $H_4^{*(i)}$  is the parity check matrix of the  $\Gamma_4^{*(i)}$ -code.

*Proof:*

Let us use representation (1) of the parity check matrix of the  $\Gamma_5^{(i)}$ -code:

$$H_5^{(i)} = \begin{bmatrix} \frac{1}{\alpha_1^i(\alpha_1^t + \alpha_1^{t-1} + 1)^i} & \cdots & \frac{1}{\alpha_{n_5}^i(\alpha_{n_5}^t + \alpha_{n_5}^{t-1} + 1)^i} \\ \frac{1}{\alpha_1^{i-1}(\alpha_1^t + \alpha_1^{t-1} + 1)^i} & \cdots & \frac{1}{\alpha_{n_5}^{i-1}(\alpha_{n_5}^t + \alpha_{n_5}^{t-1} + 1)^i} \\ \vdots & \cdots & \vdots \\ \frac{1}{\alpha_1(\alpha_1^t + \alpha_1^{t-1} + 1)^i} & \cdots & \frac{1}{\alpha_{n_5}(\alpha_{n_5}^t + \alpha_{n_5}^{t-1} + 1)^i} \\ \frac{1}{(\alpha_1^t + \alpha_1^{t-1} + 1)^i} & \cdots & \frac{1}{(\alpha_{n_5}^t + \alpha_{n_5}^{t-1} + 1)^i} \\ \vdots & \cdots & \vdots \\ \frac{\alpha_1^{(i-1)t-1}}{(\alpha_1^t + \alpha_1^{t-1} + 1)^i} & \cdots & \frac{\alpha_{n_5}^{(i-1)t-1}}{(\alpha_{n_5}^t + \alpha_{n_5}^{t-1} + 1)^i} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\alpha_1^i(\alpha_1^t + \alpha_1^{t-1} + 1)^i} & \cdots & \frac{1}{\alpha_{n_5}^i(\alpha_{n_5}^t + \alpha_{n_5}^{t-1} + 1)^i} \\ \frac{1}{\alpha_1^{i-1}(\alpha_1^t + \alpha_1^{t-1} + 1)^i} & \cdots & \frac{1}{\alpha_{n_5}^{i-1}(\alpha_{n_5}^t + \alpha_{n_5}^{t-1} + 1)^i} \\ \vdots & \cdots & \vdots \\ \frac{1}{\alpha_1(\alpha_1^t + \alpha_1^{t-1} + 1)^i} & \cdots & \frac{1}{\alpha_{n_5}(\alpha_{n_5}^t + \alpha_{n_5}^{t-1} + 1)^i} \end{bmatrix} H_4^{*(i)} \quad (6)$$

It is easy to see that in case of component-wise raising of any row

$$\left[ \frac{1}{\alpha_1^j(\alpha_1^t + \alpha_1^{t-1} + 1)^i} \cdots \frac{1}{\alpha_{n_5}^j(\alpha_{n_5}^t + \alpha_{n_5}^{t-1} + 1)^i} \right],$$

$$0 < j < i < q - 1$$

of the matrix  $H_5^{(i)}$  to the power  $t$  we obtain the corresponding row of the submatrix  $H_4^{*(i)}$ .

$$\left[ \frac{1}{\alpha_1^j(\alpha_1^t + \alpha_1^{t-1} + 1)^i} \cdots \frac{1}{\alpha_{n_5}^j(\alpha_{n_5}^t + \alpha_{n_5}^{t-1} + 1)^i} \right]^t =$$

$$\left[ \frac{1}{\alpha_1^{jt}(\alpha_1^t + \alpha_1^{t-1} + 1)^i} \cdots \frac{1}{\alpha_{n_5}^{jt}(\alpha_{n_5}^t + \alpha_{n_5}^{t-1} + 1)^i} \right] =$$

$$\left[ \frac{\alpha_1^{t(i-j)-i}}{(\alpha_1^t + \alpha_1^{t-1} + 1)^i} \cdots \frac{\alpha_{n_5}^{t(i-j)-i}}{(\alpha_{n_5}^t + \alpha_{n_5}^{t-1} + 1)^i} \right].$$

Thus, the parity check matrix of the  $\Gamma_5^{(i)}$ -code can be rewritten as

$$H_5^{(i)} = \begin{bmatrix} \frac{1}{\alpha_1^i(\alpha_1^t + \alpha_1^{t-1} + 1)^i} & \cdots & \frac{1}{\alpha_{n_5}^i(\alpha_{n_5}^t + \alpha_{n_5}^{t-1} + 1)^i} \\ & & H_4^{*(i)} \end{bmatrix}, \quad (7)$$

where the first row consists of elements of the  $GF(t)$  only:

$$\left[ \frac{1}{\alpha_1^i(\alpha_1^t + \alpha_1^{t-1} + 1)^i} \cdots \frac{1}{\alpha_{n_5}^i(\alpha_{n_5}^t + \alpha_{n_5}^{t-1} + 1)^i} \right] =$$

$$\left[ \frac{1}{(\alpha_1^{t+1} + \alpha_1^t + \alpha_1)^i} \cdots \frac{1}{(\alpha_{n_5}^{t+1} + \alpha_{n_5}^t + \alpha_{n_5})^i} \right] =$$

$$\left[ \frac{1}{G_5^{(i)}(\alpha_{i_{n_5}})} \cdots \frac{1}{G_5^{(i)}(\alpha_{i_{n_5}})} \right].$$

Hence, the redundancy of the  $\Gamma_5^{(i)}$ -code is determined by the inequality  $r_5^{(i)} \leq r_4^{*(i)} + l$ . ■

$i = q - 1, i = q$  : In accordance with the previously obtained Corollary 1 and Corollary 2 we have that the  $\Gamma_5^{(q-1)}$ -code is equivalent to the  $\Gamma_5^{(q)}$ -code and the  $\Gamma_4^{*(q-1)}$ -code is equivalent to the  $\Gamma_4^{*(q)}$ -code. Let us prove now the equivalence of the codes  $\Gamma_5^{(q-1)}$  and  $\Gamma_4^{*(q-1)}$ .

**Lemma 7:** The parity check matrices of the  $\Gamma_5^{(q-1)}$ -code and  $\Gamma_4^{*(q-1)}$ -code are equal:

$$H_5^{(q-1)} = H_4^{*(q-1)}$$

*Proof:*

Using Lemma 6, we write a relation between parity check matrices of the codes  $\Gamma_5^{(q-1)}$  and  $\Gamma_4^{*(q-1)}$ :

$$H_5^{(q-1)} = \begin{bmatrix} \frac{1}{\alpha_1^{q-1}(\alpha_1^t + \alpha_1^{t-1} + 1)^{q-1}} & \cdots & \frac{1}{\alpha_{n_5}^{q-1}(\alpha_{n_5}^t + \alpha_{n_5}^{t-1} + 1)^{q-1}} \\ & & H_4^{*(q-1)} \end{bmatrix}.$$

As the codes  $\Gamma_4^{*(q-1)}$  and  $\Gamma_4^{*(q)}$  are equivalent then their parity check matrices are equal:  $H_4^{*(q-1)} = H_4^{*(q)}$ . We show that the row

$$\left[ \frac{1}{\alpha_1^{q-1}(\alpha_1^t + \alpha_1^{t-1} + 1)^{q-1}} \cdots \frac{1}{\alpha_{n_5}^{q-1}(\alpha_{n_5}^t + \alpha_{n_5}^{t-1} + 1)^{q-1}} \right]$$

of the matrix  $H_5^{(q-1)}$  can be obtained as a linear combination of the matrix  $H_4^{*(q)}$  row:

$$\left[ \frac{\alpha_1^{t-q}}{(\alpha_1^t + \alpha_1^{t-1} + 1)^q} \cdots \frac{\alpha_{n_5}^{t-q}}{(\alpha_{n_5}^t + \alpha_{n_5}^{t-1} + 1)^q} \right],$$

its  $t$ -th power:

$$\left( \left[ \frac{\alpha_1^{t-q}}{(\alpha_1^t + \alpha_1^{t-1} + 1)^q} \cdots \frac{\alpha_{n_5}^{t-q}}{(\alpha_{n_5}^t + \alpha_{n_5}^{t-1} + 1)^q} \right] \right)^t =$$

$$\left[ \frac{1}{\alpha_1^{q-1}(\alpha_1^t + \alpha_1^{t-1} + 1)^q} \cdots \frac{1}{\alpha_{n_5}^{q-1}(\alpha_{n_5}^t + \alpha_{n_5}^{t-1} + 1)^q} \right]$$

and the matrix  $H_4^{*(q)}$  row:

$$\left[ \frac{\alpha_1^{t-q+1}}{(\alpha_1^t + \alpha_1^{t-1} + 1)^q} \cdots \frac{\alpha_{n_5}^{t-q+1}}{(\alpha_{n_5}^t + \alpha_{n_5}^{t-1} + 1)^q} \right].$$

Indeed,

$$\left[ \frac{\alpha_1^{t-q+1}}{(\alpha_1^t + \alpha_1^{t-1} + 1)^q} \cdots \frac{\alpha_{n_5}^{t-q+1}}{(\alpha_{n_5}^t + \alpha_{n_5}^{t-1} + 1)^q} \right] +$$

$$\left[ \frac{1}{\alpha_1^{q-1}(\alpha_1^t + \alpha_1^{t-1} + 1)^q} \cdots \frac{1}{\alpha_{n_5}^{q-1}(\alpha_{n_5}^t + \alpha_{n_5}^{t-1} + 1)^q} \right] +$$

$$\left[ \frac{\alpha_1^{t-q}}{(\alpha_1^t + \alpha_1^{t-1} + 1)^q} \cdots \frac{\alpha_{n_5}^{t-q}}{(\alpha_{n_5}^t + \alpha_{n_5}^{t-1} + 1)^q} \right] =$$

$$\left[ \frac{\alpha_1^t + \alpha_1^{t-1} + 1}{\alpha_1^{q-1}(\alpha_1^t + \alpha_1^{t-1} + 1)^q} \cdots \frac{\alpha_{n_5}^t + \alpha_{n_5}^{t-1} + 1}{\alpha_{n_5}^{q-1}(\alpha_{n_5}^t + \alpha_{n_5}^{t-1} + 1)^q} \right] =$$

$$\left[ \frac{1}{\alpha_1^{q-1}(\alpha_1^t + \alpha_1^{t-1} + 1)^{q-1}} \cdots \frac{1}{\alpha_{n_5}^{q-1}(\alpha_{n_5}^t + \alpha_{n_5}^{t-1} + 1)^{q-1}} \right]$$

Hence, the redundancy of the  $\Gamma_5^{(q-1)}$ -code is equal to that of the  $\Gamma_4^{*(q-1)}$ -code: ■

$$r_5^{(q-1)} = r_4^{*(q-1)}.$$

**F. Relation between the cumulative-separable codes  $\Gamma_5^{(i)}$  and  $\Gamma_6^{(i)}$**

We define the  $\Gamma_6^{(i)}$ -code as:

$$\Gamma_6^{(i)} = \Gamma(L_6, G_6^{(i)}) \text{ with } L_6 = \{GF(t^2) \setminus \{\alpha : G_6(\alpha) = 0\}\},$$

$$n_6^{(i)} = n_6 \text{ and } G_6^{(i)}(x) = (x^{t+1} + 1)^i$$

It is clear that the codes  $\Gamma_6^{(i)}$  and  $\Gamma_5^{(i)}$  are equivalent for all  $i \geq 1$  up to the permutation

$\beta \rightarrow \gamma\beta - 1$ , where  $\gamma \in GF(t^2)$  and  $\gamma^{t+1} + 1 = 0$ ,  $\beta \in L_5$ .

and therefore,  $n_6 = n_5$ ,  $k_6^{(i)} = k_5^{(i)}$  and  $d_6^{(i)} = d_5^{(i)}$ .

Thus, all the relations considered above and together with Corollary 3 allow us to present the following Table I and pattern in Fig. 1.

## V. MINIMUM DISTANCE OF CUMULATIVE-SEPARABLE CODES

The minimum distances of primary separable codes  $\Gamma_1 - \Gamma_6$  have been determined in [4], [5]. Now let us take the  $\Gamma_6$ -code as an example to determine the minimum distances of the corresponding separable-cumulative codes  $\Gamma_6^{(i)}$  in case of the cumulativity order  $i > 1$ .

Let us consider  $q - 1$  groups of elements that form a set  $L_6$ :

1.  $L_{6(1)}^{(j)} = \left\{ \alpha^{(t-1)l + \frac{t-1}{q-1}j} \right\}_{l=0, \dots, t}; j = 0, \dots, q-2$  with the exception of  $j = j^*$ , such that

$$G_6(\alpha^{(t-1)l + \frac{t-1}{q-1}j^*}) = \alpha^{\frac{(t-1)(t+1)}{q-1}j^*} + 1 = 0.$$

In other words, we exclude  $j = j^*$  such that

$$\alpha^{\frac{(t-1)(t+1)}{q-1}j^*} = q - 1, \alpha \text{ is a primitive element of } GF(t^2).$$

It is easy to show that

$$L_{6(1)}^{(i)} \cap L_{6(1)}^{(j)} = \emptyset \text{ for any } i \neq j; i, j = 0, \dots, q-2.$$

Therefore the number of different groups  $L_{6(1)}^{(j)}$  in  $L_6$  is  $q-2$ . Further, we shall denote vector  $\mathbf{a}$  values in the set  $L_{6(1)}^{(j)}$  positions by

$$a_{j,l,1}, \quad j = 0, 1, \dots, j^* - 1, j^* + 1, \dots, q-2; \\ l = 0, \dots, t.$$

2.  $L_{6(0)} = \{0\}$ . The vector  $\mathbf{a}$  value in this position is denoted by  $a_0$ .

It should be noted that the value of the polynomial  $G_6(x)$  for all elements from one numerator group is the same and it is equal to:

1. For elements from the numerator group  $L_{6(1)}^{(j)}$ :

$$G_6^{(i)}(\alpha^{(t-1)l + \frac{t-1}{q-1}j}) = \left( (\alpha^{(t-1)l + \frac{t-1}{q-1}j})^{t+1} + 1 \right)^i = \left( \alpha^{\frac{(t-1)(t+1)}{q-1}j} + 1 \right)^i.$$

Obviously, all values

$$G_6^{(i)}(\alpha^{(t-1)l + \frac{t-1}{q-1}j}) \in GF(q), \\ j = 0, 1, \dots, j^* - 1, j^* + 1, \dots, q-2.$$

2. For elements from the numerator group  $L_{6(0)}$ :  $G_6^{(i)}(0) = 1$ .

Let us consider now the submatrix  $H_{6(1,j)}^{(i)}$  of the parity check matrix  $H_6^{(i)}$ . It corresponds to the numerator group  $L_{6(1)}^{(j)}$

that we determined above. Here we use representation (2):

$$H_{6(1,j)}^{(i)} = \begin{bmatrix} \frac{1}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^t} & \cdots & \frac{1}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^t} \\ \frac{\alpha^{\frac{t-1}{q-1}j}}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^t} & \cdots & \frac{\alpha^{\frac{t-1}{q-1}j+(t-1)t}}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^t} \\ \vdots & \cdots & \vdots \\ \frac{\left( \alpha^{\frac{t-1}{q-1}j} \right)^t}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^t} & \cdots & \frac{\left( \alpha^{\frac{t-1}{q-1}j+(t-1)t} \right)^t}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^t} \\ \frac{1}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^{i-1}} & \cdots & \frac{1}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^{i-1}} \\ \frac{\alpha^{\frac{t-1}{q-1}j}}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^{i-1}} & \cdots & \frac{\alpha^{\frac{t-1}{q-1}j+(t-1)t}}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^{i-1}} \\ \vdots & \cdots & \vdots \\ \frac{\left( \alpha^{\frac{t-1}{q-1}j} \right)^t}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^2} & \cdots & \frac{\left( \alpha^{\frac{t-1}{q-1}j+(t-1)t} \right)^t}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^2} \\ \frac{1}{\alpha^{\frac{(t-1)(t+1)}{q-1}j+1}} & \cdots & \frac{1}{\alpha^{\frac{(t-1)(t+1)}{q-1}j+1}} \\ \vdots & \cdots & \vdots \\ \frac{\left( \alpha^{\frac{t-1}{q-1}j} \right)^t}{\alpha^{\frac{(t-1)(t+1)}{q-1}j+1}} & \cdots & \frac{\left( \alpha^{\frac{t-1}{q-1}j+(t-1)t} \right)^t}{\alpha^{\frac{(t-1)(t+1)}{q-1}j+1}} \end{bmatrix}. \quad (8)$$

Let us take the matrix column corresponding to the position  $\{0\}$ :

$$H_{6(0)}^{(i)} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}. \quad (9)$$

Now we write the parity check matrix of the  $\Gamma_6^{(i)}$ -code by using the submatrices  $H_{6(1,j)}^{(i)}$  and  $H_{6(0)}^{(i)}$ :

$$H_6^{(i)} = \begin{bmatrix} H_{6(1,0)}^{(i)} & H_{6(1,1)}^{(i)} & \cdots & H_{6(1,q-2)}^{(i)} & H_{6(0)}^{(i)} \end{bmatrix}.$$

Let us choose the vector  $\mathbf{a}$  of the following form:

$$\mathbf{a} = \begin{bmatrix} a_{0,0,1} a_{0,1,1} \dots a_{0,t,1} & a_{1,0,1} a_{1,1,1} \dots a_{1,t,1} & \dots \\ a_{q-2,0,1} a_{q-2,1,1} \dots a_{q-2,t,1} & a_0 & \dots \end{bmatrix}.$$

Assume that

$$a_{0,0,1} = a_{0,1,1} = \dots = a_{0,t,1} = a_{0,1} \in GF(q); \\ a_{1,0,1} = a_{1,1,1} = \dots = a_{1,t,1} = a_{1,1} \in GF(q);$$

...

$$a_{q-2,0,1} = a_{q-2,1,1} = \dots = a_{q-2,t,1} = a_{q-2,t,1} \in GF(q).$$

Now we calculate the result of multiplication of the vector  $\mathbf{a}$

by the parity check matrix of the code:

$$\begin{aligned}
 \mathbf{a} \cdot H_6^{(i)T} &= [a_{0,1} \dots a_{0,1} \dots a_{q-2,1} \dots a_{q-2,1} a_0] \cdot \\
 &\left[ H_{6(1,0)}^{(i)} \dots H_{6(1,q-2)}^{(i)} H_{6(0)}^{(i)} \right]^T = \\
 &[a_{0,1} \dots a_{0,1}] \cdot \left[ H_{6(1,0)}^{(i)} \right]^T + \dots \\
 &+ [a_{q-2,1} \dots a_{q-2,1}] \cdot \left[ H_{6(1,q-2)}^{(i)} \right]^T + \\
 &a_0 \cdot \left[ H_{6(0)}^{(i)} \right]^T = a_{0,1} [1 \dots 1] \cdot \left[ H_{6(1,0)}^{(i)} \right]^T + \dots + \\
 &a_{q-2,1} [1 \dots 1] \cdot \left[ H_{6(1,q-2)}^{(i)} \right]^T + a_0 \cdot \left[ H_{6(0)}^{(i)} \right]^T.
 \end{aligned} \tag{10}$$

Let us calculate the value of the row vector that is obtained from multiplying the identity vector by the submatrix  $H_{6(1,j)}^{(i)}$  (8), for all  $j$  from 0 to  $q-2$  with an exception  $j = j^*$ :  $\alpha^{\frac{(t-1)(t+1)}{q-1}j^*} = q-1$ :

$$[1 \dots 1] \cdot \begin{bmatrix} \frac{1}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^t} } & \dots & \frac{1}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^t} } \\ \frac{\alpha^{\frac{t-1}{q-1}j}}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^t} } & \dots & \frac{\alpha^{\frac{t-1}{q-1}j+(t-1)t}}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^t} } \\ \dots & \dots & \dots \\ \frac{\left( \alpha^{\frac{t-1}{q-1}j} \right)^t}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^t} } & \dots & \frac{\left( \alpha^{\frac{t-1}{q-1}j+(t-1)t} \right)^t}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^t} } \\ \frac{1}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^{t-1}} } & \dots & \frac{1}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^{t-1}} } \\ \frac{\alpha^{\frac{t-1}{q-1}j}}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^{t-1}} } & \dots & \frac{\alpha^{\frac{t-1}{q-1}j+(t-1)t}}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^{t-1}} } \\ \dots & \dots & \dots \\ \frac{\left( \alpha^{\frac{t-1}{q-1}j} \right)^t}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^2} } & \dots & \frac{\left( \alpha^{\frac{t-1}{q-1}j+(t-1)t} \right)^t}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^2} } \\ \frac{1}{\alpha^{\frac{(t-1)(t+1)}{q-1}j+1}} & \dots & \frac{1}{\alpha^{\frac{(t-1)(t+1)}{q-1}j+1}} \\ \dots & \dots & \dots \\ \frac{\left( \alpha^{\frac{t-1}{q-1}j} \right)^t}{\alpha^{\frac{(t-1)(t+1)}{q-1}j+1}} & \dots & \frac{\left( \alpha^{\frac{t-1}{q-1}j+(t-1)t} \right)^t}{\alpha^{\frac{(t-1)(t+1)}{q-1}j+1}} \end{bmatrix}^T$$

$$= \begin{bmatrix} \frac{\frac{1+1+\dots+1}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^t} }}{\alpha^{\frac{t-1}{q-1}j} + \alpha^{\frac{t-1}{q-1}j+(t-1)} + \dots + \alpha^{\frac{t-1}{q-1}j+(t-1)t}}}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^t} } \\ \vdots \\ \frac{\left( \alpha^{\frac{t-1}{q-1}j} \right)^t + \left( \alpha^{\frac{t-1}{q-1}j+(t-1)} \right)^t + \dots + \left( \alpha^{\frac{t-1}{q-1}j+(t-1)t} \right)^t}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^t} } \\ \frac{\frac{1+1+\dots+1}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^{t-1}} }}{\alpha^{\frac{t-1}{q-1}j} + \alpha^{\frac{t-1}{q-1}j+(t-1)} + \dots + \alpha^{\frac{t-1}{q-1}j+(t-1)t}}}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^{t-1}} } \\ \vdots \\ \frac{\left( \alpha^{\frac{t-1}{q-1}j} \right)^t + \left( \alpha^{\frac{t-1}{q-1}j+(t-1)} \right)^t + \dots + \left( \alpha^{\frac{t-1}{q-1}j+(t-1)t} \right)^t}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^2} } \\ \frac{\frac{1+1+\dots+1}{\alpha^{\frac{(t-1)(t+1)}{q-1}j+1}}}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^2} } \\ \vdots \\ \frac{\left( \alpha^{\frac{t-1}{q-1}j} \right)^t + \left( \alpha^{\frac{t-1}{q-1}j+(t-1)} \right)^t + \dots + \left( \alpha^{\frac{t-1}{q-1}j+(t-1)t} \right)^t}{\alpha^{\frac{(t-1)(t+1)}{q-1}j+1}} \end{bmatrix}^T =$$

$$= \begin{bmatrix} \frac{t+1}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^t} } \\ \frac{\alpha^{\frac{t-1}{q-1}j}}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^t} } (1 + \alpha^{t-1} + \dots + \alpha^{(t-1)t}) \\ \vdots \\ \frac{\left( \alpha^{\frac{t-1}{q-1}j} \right)^t}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^t} } (1 + (\alpha^{t-1})^t + \dots + (\alpha^{(t-1)t})^t) \\ \frac{t+1}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^{t-1}} } \\ \frac{\alpha^{\frac{t-1}{q-1}j}}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^{t-1}} } (1 + \alpha^{t-1} + \dots + \alpha^{(t-1)t}) \\ \vdots \\ \frac{\left( \alpha^{\frac{t-1}{q-1}j} \right)^t}{\left( \alpha^{\frac{(t-1)(t+1)}{q-1}j+1 \right)^2} } (1 + (\alpha^{t-1})^t + \dots + (\alpha^{(t-1)t})^t) \\ \frac{t+1}{\alpha^{\frac{(t-1)(t+1)}{q-1}j+1}} \\ \vdots \\ \frac{\left( \alpha^{\frac{t-1}{q-1}j} \right)^t}{\alpha^{\frac{(t-1)(t+1)}{q-1}j+1}} (1 + (\alpha^{t-1})^t + \dots + (\alpha^{(t-1)t})^t) \end{bmatrix}^T$$

Since  $1, \alpha^{t-1}, \dots, \alpha^{(t-1)t}$  are all the roots of the polynomial  $x^{t+1} - 1$ , then the sum of any nonzero powers of these elements is always equal to zero:

$$\sum_{i=0}^t \alpha^{(t-1)i} = 0.$$



Besides  $t + 1 = 1 \pmod{q}$ , and we obtain:

$$[1 \dots 11] \cdot H_6^{(i)T} = \begin{bmatrix} \frac{1}{\left(\alpha^{\frac{(t-1)(t+1)}{q-1}j+1\right)^t} \\ 0 \\ \vdots \\ 0 \\ \frac{1}{\left(\alpha^{\frac{(t-1)(t+1)}{q-1}j+1\right)^{t-1}} \\ 0 \\ \vdots \\ 0 \\ \frac{1}{\alpha^{\frac{(t-1)(t+1)}{q-1}j+1}} \\ \vdots \\ 0 \end{bmatrix}^T$$

Therefore expression (10) can be rewritten as:

$$\mathbf{a}^* \cdot H_6^{*(i)T} = \begin{bmatrix} a_{0,1} & \dots & a_{q-2,1} & a_0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2^t} & \dots & \frac{1}{\left(\alpha^{\frac{(t-1)(t+1)}{q-1}t+1\right)^t} & 1 \\ \frac{1}{2^{t-1}} & \dots & \frac{1}{\left(\alpha^{\frac{(t-1)(t+1)}{q-1}t+1\right)^{t-1}} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{2} & \dots & \frac{1}{\alpha^{\frac{(t-1)(t+1)}{q-1}t+1}} & 1 \end{bmatrix}^T, \quad (11)$$

where the length of vector  $\mathbf{a}^*$  is equal to  $q - 1$ .

It is easy to note that all  $q - 1$  different nonzero elements of the field  $GF(q)$  are in the last row of the matrix. The matrix in (11) itself is the Vandermonde matrix. These remarks permit us to formulate the following theorem on the minimum distance of cumulative-separable codes from the  $\Gamma_6^{(i)}$  subclass of codes.

**Theorem 1:** The minimum distance of the codes from the subclass  $\Gamma_6^{(i)}$  is equal to  $i(t + 1) + 1$  for  $1 < i < q - 1$ .

*Proof:*

First of all, it should be noted that the matrix  $H_6^{*(i)}$  in (11) is the parity check matrix of RS-code. It means that for  $i < q - 1$  there always exists a codeword of RS-code with nonzero symbol on the position  $\{0\}$  and with other  $i$  nonzero symbols in positions that correspond to the other columns of the parity check matrix  $H_6^{*(i)}$  in (11). Using expression (10) the following value of the minimum distance of the  $\Gamma_6^{(i)}$ -code can be obtained for  $1 < i < q - 1$ :

$$d_6^{(i)} = i(t + 1) + 1 = \deg G_6^{(i)}(x) + 1. \quad \blacksquare$$

## VI. DIMENSION OF THE CUMULATIVE-SEPARABLE CODES THAT ARE OBTAINED FROM THE $\Gamma_6$ -CODE

According to the definition of the  $\Gamma_6$ -code the cumulative-separable  $\Gamma(L_6, G_6^{(q)})$ -code corresponding to it is given by the polynomial  $G_6^{(q)}(x) = (x^{t+1} + 1)^q$  and set  $L_6 = \{GF(t^2) \setminus \{\alpha : G_6(\alpha) = 0\}\}$ , where  $t = q^l$ . The parity check matrix  $H_6$  of the  $\Gamma(L_6, G_6^{(q)})$ -code can be presented as set of submatrices  $h_1, h_2, \dots, h_{q-1}$  by using expression (2). Here

for simplicity we denote the length of the  $\Gamma_6$ -code by  $n$ .

**1.** Consider subsets of submatrix  $h_1$  rows that can be obtained by transformations of other rows of this submatrix:

$$h_1 = \begin{bmatrix} \frac{1}{\alpha_1^{t+1}+1} & \frac{1}{\alpha_2^{t+1}+1} & \dots & \frac{1}{\alpha_{n-1}^{t+1}+1} & 1 \\ \frac{\alpha_1}{\alpha_1^{t+1}+1} & \frac{\alpha_2}{\alpha_2^{t+1}+1} & \dots & \frac{\alpha_{n-1}}{\alpha_{n-1}^{t+1}+1} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\alpha_1^t}{\alpha_1^{t+1}+1} & \frac{\alpha_2^t}{\alpha_2^{t+1}+1} & \dots & \frac{\alpha_{n-1}^t}{\alpha_{n-1}^{t+1}+1} & 0 \end{bmatrix}, \quad (12)$$

where  $\alpha_i \in L_6$  and  $\alpha_n = 0$ .

**1.1.** All elements of the first row in the submatrix  $h_1$  are elements of the field  $GF(t)$ . Indeed,

$$\left(\frac{1}{\alpha_i^{t+1}+1}\right)^t = \frac{1}{\alpha_i^{t+1}+1}$$

for any  $i = 1, \dots, n - 1$ .

**1.2.** The row

$$\begin{bmatrix} \frac{\alpha_1^t}{\alpha_1^{t+1}+1} & \frac{\alpha_2^t}{\alpha_2^{t+1}+1} & \dots & \frac{\alpha_{n-1}^t}{\alpha_{n-1}^{t+1}+1} & 0 \end{bmatrix}$$

is obtained by element-wise raising to the power  $t$  of the row

$$\begin{bmatrix} \frac{\alpha_1}{\alpha_1^{t+1}+1} & \frac{\alpha_2}{\alpha_2^{t+1}+1} & \dots & \frac{\alpha_{n-1}}{\alpha_{n-1}^{t+1}+1} & 0 \end{bmatrix}$$

of the submatrix.

Therefore, the number of  $q$ -ary rows equal to  $\delta_1 = l + 2l$  can be deleted from the submatrix  $h_1$  as they are linearly dependent on the other rows of this submatrix.

**2.** Consider now a submatrix  $h_2$ :

$$h_2 = \begin{bmatrix} \frac{1}{(\alpha_1^{t+1}+1)^2} & \frac{1}{(\alpha_2^{t+1}+1)^2} & \dots & \frac{1}{(\alpha_{n-1}^{t+1}+1)^2} & 1 \\ \frac{\alpha_1}{(\alpha_1^{t+1}+1)^2} & \frac{\alpha_2}{(\alpha_2^{t+1}+1)^2} & \dots & \frac{\alpha_{n-1}}{(\alpha_{n-1}^{t+1}+1)^2} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\alpha_1^t}{(\alpha_1^{t+1}+1)^2} & \frac{\alpha_2^t}{(\alpha_2^{t+1}+1)^2} & \dots & \frac{\alpha_{n-1}^t}{(\alpha_{n-1}^{t+1}+1)^2} & 0 \end{bmatrix}, \quad (13)$$

where  $\alpha_i \in L_6$  and  $\alpha_n = 0$ .

Let us determine a set of linearly dependent  $q$ -ary rows in this submatrix.

**2.1.** All elements of the first row of the submatrix  $h_2$  are elements of the field  $GF(t)$ . Indeed,

$$\left(\frac{1}{(\alpha_i^{t+1}+1)^2}\right)^t = \frac{1}{(\alpha_i^{t+1}+1)^2}$$

for any  $i = 1, \dots, n - 1$ .

**2.2.** The row

$$\begin{bmatrix} \frac{\alpha_1^t}{(\alpha_1^{t+1}+1)^2} & \frac{\alpha_2^t}{(\alpha_2^{t+1}+1)^2} & \dots & \frac{\alpha_{n-1}^t}{(\alpha_{n-1}^{t+1}+1)^2} & 0 \end{bmatrix}$$

is obtained by the element-by-element raising of the row of the submatrix

$$\begin{bmatrix} \frac{\alpha_1}{(\alpha_1^{t+1}+1)^2} & \frac{\alpha_2}{(\alpha_2^{t+1}+1)^2} & \dots & \frac{\alpha_{n-1}}{(\alpha_{n-1}^{t+1}+1)^2} & 0 \end{bmatrix}$$

to the power  $t$ .

**2.3.** The row

$$\begin{bmatrix} \frac{\alpha_1^{t-1}}{(\alpha_1^{t+1}+1)^2} & \frac{\alpha_2^{t-1}}{(\alpha_2^{t+1}+1)^2} & \dots & \frac{\alpha_{n-1}^{t-1}}{(\alpha_{n-1}^{t+1}+1)^2} & 0 \end{bmatrix}$$

is obtained as a linear combination of the corresponding row of the submatrix  $h_1$

$$\begin{bmatrix} \frac{\alpha_1^{t-1}}{\alpha_1^{t+1}+1} & \frac{\alpha_2^{t-1}}{\alpha_2^{t+1}+1} & \dots & \frac{\alpha_{n-1}^{t-1}}{\alpha_{n-1}^{t+1}+1} & 0 \end{bmatrix}$$

and the  $t$ -th power of the corresponding row of submatrix  $h_2$

$$\begin{bmatrix} \frac{\alpha_1^2}{(\alpha_1^{t+1}+1)^2} & \frac{\alpha_2^2}{(\alpha_2^{t+1}+1)^2} & \cdots & \frac{\alpha_{n-1}^2}{(\alpha_{n-1}^{t+1}+1)^2} & 0 \end{bmatrix}.$$

It means that we can write the following relation for any element of these rows:

$$\frac{\alpha_i^{t-1}}{(\alpha_i^{t+1}+1)^2} = \frac{\alpha_i^{t-1}}{(\alpha_i^{t+1}+1)} - \left( \frac{\alpha_i^2}{(\alpha_i^{t+1}+1)^2} \right)^t$$

for all  $i = 1, \dots, n-1$ .

Thus, the number of  $q$ -ary rows equal to  $\delta_2 = l + 2l + 2l$  can be deleted from  $h_2$  as they are linearly dependent on the other rows of this and preceding submatrix.

*Remark 1:* Note that we used the row:

$$\begin{bmatrix} \frac{\alpha_1^{t-1}}{\alpha_1^{t+1}+1} & \frac{\alpha_2^{t-1}}{\alpha_2^{t+1}+1} & \cdots & \frac{\alpha_{n-1}^{t-1}}{\alpha_{n-1}^{t+1}+1} & 0 \end{bmatrix}$$

of the submatrix  $h_1$  when representing the row of the submatrix  $h_2$ .

**3.** Consider the submatrix  $h_3$ :

$$h_3 = \begin{bmatrix} \frac{1}{(\alpha_1^{t+1}+1)^3} & \frac{1}{(\alpha_2^{t+1}+1)^3} & \cdots & \frac{1}{(\alpha_{n-1}^{t+1}+1)^3} & 1 \\ \frac{\alpha_1}{(\alpha_1^{t+1}+1)^3} & \frac{\alpha_2}{(\alpha_2^{t+1}+1)^3} & \cdots & \frac{\alpha_{n-1}}{(\alpha_{n-1}^{t+1}+1)^3} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\alpha_1^t}{(\alpha_1^{t+1}+1)^3} & \frac{\alpha_2^t}{(\alpha_2^{t+1}+1)^3} & \cdots & \frac{\alpha_{n-1}^t}{(\alpha_{n-1}^{t+1}+1)^3} & 0 \end{bmatrix}, \quad (14)$$

where  $\alpha_i \in L_6$  and  $\alpha_n = 0$ .

Let us determine a set of linearly dependent  $q$ -ary rows in this submatrix.

**3.1.** All elements of the first row of the submatrix  $h_3$  are elements of the field  $GF(t)$ . Indeed,

$$\left( \frac{1}{(\alpha_i^{t+1}+1)^3} \right)^t = \frac{1}{(\alpha_i^{t+1}+1)^3} \text{ for any } i = 1, \dots, n-1.$$

**3.2.** The row

$$\begin{bmatrix} \frac{\alpha_1^t}{(\alpha_1^{t+1}+1)^3} & \frac{\alpha_2^t}{(\alpha_2^{t+1}+1)^3} & \cdots & \frac{\alpha_{n-1}^t}{(\alpha_{n-1}^{t+1}+1)^3} & 0 \end{bmatrix}$$

is obtained by element-by-element raising of the row of the submatrix

$$\begin{bmatrix} \frac{\alpha_1}{(\alpha_1^{t+1}+1)^3} & \frac{\alpha_2}{(\alpha_2^{t+1}+1)^3} & \cdots & \frac{\alpha_{n-1}}{(\alpha_{n-1}^{t+1}+1)^3} & 0 \end{bmatrix}$$

to the  $t$ -th power.

**3.3.** The row

$$\begin{bmatrix} \frac{\alpha_1^{t-1}}{(\alpha_1^{t+1}+1)^3} & \frac{\alpha_2^{t-1}}{(\alpha_2^{t+1}+1)^3} & \cdots & \frac{\alpha_{n-1}^{t-1}}{(\alpha_{n-1}^{t+1}+1)^3} & 0 \end{bmatrix}$$

is obtained as a linear combination of the corresponding submatrix  $h_2$  row

$$\begin{bmatrix} \frac{\alpha_1^{t-1}}{(\alpha_1^{t+1}+1)^2} & \frac{\alpha_2^{t-1}}{(\alpha_2^{t+1}+1)^2} & \cdots & \frac{\alpha_{n-1}^{t-1}}{(\alpha_{n-1}^{t+1}+1)^2} & 0 \end{bmatrix}$$

and the  $t$ -th power of the corresponding row of the submatrix  $h_3$

$$\begin{bmatrix} \frac{\alpha_1^2}{(\alpha_1^{t+1}+1)^3} & \frac{\alpha_2^2}{(\alpha_2^{t+1}+1)^3} & \cdots & \frac{\alpha_{n-1}^2}{(\alpha_{n-1}^{t+1}+1)^3} & 0 \end{bmatrix}.$$

It means that the following relation:

$$\frac{\alpha_i^{t-1}}{(\alpha_i^{t+1}+1)^3} = \frac{\alpha_i^{t-1}}{(\alpha_i^{t+1}+1)^2} - \left( \frac{\alpha_i^2}{(\alpha_i^{t+1}+1)^3} \right)^t$$

can be written for any element of these rows for all  $i = 1, \dots, n-1$ .

**3.4.** The row

$$\begin{bmatrix} \frac{\alpha_1^{t-2}}{(\alpha_1^{t+1}+1)^3} & \frac{\alpha_2^{t-2}}{(\alpha_2^{t+1}+1)^3} & \cdots & \frac{\alpha_{n-1}^{t-2}}{(\alpha_{n-1}^{t+1}+1)^3} & 0 \end{bmatrix}$$

is obtained as a linear combination of the corresponding row of the submatrix  $h_1$

$$\begin{bmatrix} \frac{\alpha_1^{t-2}}{\alpha_1^{t+1}+1} & \frac{\alpha_2^{t-2}}{\alpha_2^{t+1}+1} & \cdots & \frac{\alpha_{n-1}^{t-2}}{\alpha_{n-1}^{t+1}+1} & 0 \end{bmatrix},$$

the row of the submatrix  $h_2$

$$\begin{bmatrix} \frac{\alpha_1^{t-2}}{(\alpha_1^{t+1}+1)^2} & \frac{\alpha_2^{t-2}}{(\alpha_2^{t+1}+1)^2} & \cdots & \frac{\alpha_{n-1}^{t-2}}{(\alpha_{n-1}^{t+1}+1)^2} & 0 \end{bmatrix}$$

and the  $t$ -th power of the row

$$\begin{bmatrix} \frac{\alpha_1^3}{(\alpha_1^{t+1}+1)^3} & \frac{\alpha_2^3}{(\alpha_2^{t+1}+1)^3} & \cdots & \frac{\alpha_{n-1}^3}{(\alpha_{n-1}^{t+1}+1)^3} & 0 \end{bmatrix}$$

of the submatrix  $h_3$ . It means that we can write the following relations:

$$\begin{aligned} \frac{\alpha_i^{t-2}}{(\alpha_i^{t+1}+1)^3} &= \frac{\alpha_i^{t-2}}{(\alpha_i^{t+1}+1)^2} - \frac{\alpha_i^{2t-1}}{(\alpha_i^{t+1}+1)^3}, \\ \frac{\alpha_i^{2t-1}}{(\alpha_i^{t+1}+1)^3} &= \frac{\alpha_i^{2t-1}}{(\alpha_i^{t+1}+1)^2} - \left( \frac{\alpha_i^3}{(\alpha_i^{t+1}+1)^3} \right)^t, \\ \frac{\alpha_i^{2t-1}}{(\alpha_i^{t+1}+1)^2} &= \frac{\alpha_i^{t-2}}{(\alpha_i^{t+1}+1)} - \frac{\alpha_i^t}{(\alpha_i^{t+1}+1)^2}, \end{aligned}$$

for any element of this row for all  $i = 1, \dots, n-1$ .

Thus, the number of the  $q$ -ary row equal to  $\delta_3 = l + 2l + 2l + 2l$  can be deleted from the matrix  $h_3$  as they are linearly dependent on the other rows of this and preceding submatrices.

*Remark 2:* Note that the row of the submatrix  $h_1$

$$\begin{bmatrix} \frac{\alpha_1^{t-2}}{\alpha_1^{t+1}+1} & \frac{\alpha_2^{t-2}}{\alpha_2^{t+1}+1} & \cdots & \frac{\alpha_{n-1}^{t-2}}{\alpha_{n-1}^{t+1}+1} & 0 \end{bmatrix}$$

was used for representing the row of the submatrix  $h_3$ .

It is clear that the similar situation takes place for any submatrix.

**j.** ( $j \leq q-1$ ) Consider a submatrix  $h_j$ :

$$h_j = \begin{bmatrix} \frac{1}{(\alpha_1^{t+1}+1)^j} & \frac{1}{(\alpha_2^{t+1}+1)^j} & \cdots & \frac{1}{(\alpha_{n-1}^{t+1}+1)^j} & 1 \\ \frac{\alpha_1}{(\alpha_1^{t+1}+1)^j} & \frac{\alpha_2}{(\alpha_2^{t+1}+1)^j} & \cdots & \frac{\alpha_{n-1}}{(\alpha_{n-1}^{t+1}+1)^j} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\alpha_1^t}{(\alpha_1^{t+1}+1)^j} & \frac{\alpha_2^t}{(\alpha_2^{t+1}+1)^j} & \cdots & \frac{\alpha_{n-1}^t}{(\alpha_{n-1}^{t+1}+1)^j} & 0 \end{bmatrix}, \quad (15)$$

where  $\alpha_i \in L_6$  and  $\alpha_n = 0$ .

Let us determine a set of linearly dependent  $q$ -ary rows of this submatrix.

**j.1.** All elements of the first row of the submatrix  $h_j$  are elements of the field  $GF(t)$ . Indeed,

$$\left( \frac{1}{(\alpha_i^{t+1}+1)^j} \right)^t = \frac{1}{(\alpha_i^{t+1}+1)^j} \text{ for any } i = 1, \dots, n-1.$$

**j.2.** The row

$$\begin{bmatrix} \frac{\alpha_1^t}{(\alpha_1^{t+1}+1)^j} & \frac{\alpha_2^t}{(\alpha_2^{t+1}+1)^j} & \cdots & \frac{\alpha_{n-1}^t}{(\alpha_{n-1}^{t+1}+1)^j} & 0 \end{bmatrix}$$

is obtained by element-by-element raising of the row

$$\begin{bmatrix} \frac{\alpha_1}{(\alpha_1^{t+1}+1)^j} & \frac{\alpha_2}{(\alpha_2^{t+1}+1)^j} & \cdots & \frac{\alpha_{n-1}}{(\alpha_{n-1}^{t+1}+1)^j} & 0 \end{bmatrix}$$

of the submatrix to the  $t$ -th power.

**j.3.** The row

$$\left[ \frac{\alpha_1^{t-1}}{(\alpha_1^{t+1}+1)^j} \quad \frac{\alpha_2^{t-1}}{(\alpha_2^{t+1}+1)^j} \quad \cdots \quad \frac{\alpha_{n-1}^{t-1}}{(\alpha_{n-1}^{t+1}+1)^j} \quad 0 \right]$$

is obtained as a linear combination of corresponding row of the submatrix  $h_{j-1}$  and the  $t$ -th power of the corresponding row of the submatrix  $h_j$ . Thus, the relation

$$\frac{\alpha_i^{t-1}}{(\alpha_i^{t+1}+1)^j} = \frac{\alpha_i^{t-1}}{(\alpha_i^{t+1}+1)^{j-1}} - \left( \frac{\alpha_i^2}{(\alpha_i^{t+1}+1)^j} \right)^t$$

can be written for any element of these rows for all  $i = 1, \dots, n-1$ .

And so on.

Finally, at the last step we have the following relations for rows of the  $j$ -th submatrix:

**j.j+1.** The row

$$\left[ \frac{\alpha_1^{t-(j-1)}}{(\alpha_1^{t+1}+1)^j} \quad \frac{\alpha_2^{t-(j-1)}}{(\alpha_2^{t+1}+1)^j} \quad \cdots \quad \frac{\alpha_{n-1}^{t-(j-1)}}{(\alpha_{n-1}^{t+1}+1)^j} \quad 0 \right]$$

is obtained as linear combination of preceding rows of the submatrices  $h_j$  and  $h_{j-1}, \dots, h_2, h_1$ . In other words, we can write following relations for any elements of this row:

$$\begin{aligned} \frac{\alpha_i^{t-(j-1)}}{(\alpha_i^{t+1}+1)^j} &= \frac{\alpha_i^{t-(j-1)}}{(\alpha_i^{t+1}+1)^{j-1}} - \frac{\alpha_i^{2t-(j-2)}}{(\alpha_i^{t+1}+1)^j}, \\ \frac{\alpha_i^{2t-(j-2)}}{(\alpha_i^{t+1}+1)^j} &= \frac{\alpha_i^{2t-(j-2)}}{(\alpha_i^{t+1}+1)^{j-1}} - \frac{\alpha_i^{3t-(j-3)}}{(\alpha_i^{t+1}+1)^j}, \\ &\vdots \\ \frac{\alpha_i^{(j-1)t-1}}{(\alpha_i^{t+1}+1)^j} &= \frac{\alpha_i^{(j-1)t-1}}{(\alpha_i^{t+1}+1)^{j-1}} - \left( \frac{\alpha_i^j}{(\alpha_i^{t+1}+1)^j} \right)^t, \end{aligned}$$

for all  $i = 1, \dots, n-1$ .

Here, the row

$$\left[ \frac{\alpha_1^{t-(j-1)}}{(\alpha_1^{t+1}+1)^{j-1}} \quad \frac{\alpha_2^{t-(j-1)}}{(\alpha_2^{t+1}+1)^{j-1}} \quad \cdots \quad \frac{\alpha_{n-1}^{t-(j-1)}}{(\alpha_{n-1}^{t+1}+1)^{j-1}} \quad 0 \right]$$

belongs to the submatrix  $h_{j-1}$ , whereas the row

$$\left[ \frac{\alpha_1^j}{(\alpha_1^{t+1}+1)^j} \quad \frac{\alpha_2^j}{(\alpha_2^{t+1}+1)^j} \quad \cdots \quad \frac{\alpha_{n-1}^j}{(\alpha_{n-1}^{t+1}+1)^j} \quad 0 \right]$$

is a previous one in the submatrix  $h_j$  because  $t-(j-1) > j$ .

Let us consider now  $j-2$  rows with elements

$$\left\{ \frac{\alpha_i^{2t-(j-2)}}{(\alpha_i^{t+1}+1)^{j-1}} \right\}, \left\{ \frac{\alpha_i^{3t-(j-3)}}{(\alpha_i^{t+1}+1)^{j-1}} \right\}, \dots, \left\{ \frac{\alpha_i^{(j-1)t-1}}{(\alpha_i^{t+1}+1)^{j-1}} \right\},$$

$i = 1, \dots, n.$

The following relations can be written for the components of these rows:

$$\begin{aligned} \frac{\alpha_i^{t-(j-1)}}{(\alpha_i^{t+1}+1)^{j-1}} &= \frac{\alpha_i^{t-(j-1)}}{(\alpha_i^{t+1}+1)^{j-2}} - \frac{\alpha_i^{2t-(j-2)}}{(\alpha_i^{t+1}+1)^{j-1}}, \\ \frac{\alpha_i^{2t-(j-2)}}{(\alpha_i^{t+1}+1)^{j-1}} &= \frac{\alpha_i^{2t-(j-2)}}{(\alpha_i^{t+1}+1)^{j-2}} - \frac{\alpha_i^{3t-(j-3)}}{(\alpha_i^{t+1}+1)^{j-1}}, \\ &\vdots \\ \frac{\alpha_i^{(j-2)t-2}}{(\alpha_i^{t+1}+1)^{j-1}} &= \frac{\alpha_i^{(j-2)t-2}}{(\alpha_i^{t+1}+1)^{j-2}} - \frac{\alpha_i^{(j-1)t-1}}{(\alpha_i^{t+1}+1)^{j-1}}, \end{aligned}$$

for all  $i = 1, \dots, n$ ,  $\alpha_n = 0$ .

Here the rows

$$\left[ \left( \frac{\alpha_1^{t-(j-1)}}{(\alpha_1^{t+1}+1)^{j-1}} \quad \frac{\alpha_2^{t-(j-1)}}{(\alpha_2^{t+1}+1)^{j-1}} \quad \cdots \quad \frac{\alpha_{n-1}^{t-(j-1)}}{(\alpha_{n-1}^{t+1}+1)^{j-1}} \quad 0 \right) \right]$$

and

$$\left[ \frac{\alpha_1^{t-(j-1)}}{(\alpha_1^{t+1}+1)^{j-2}} \quad \frac{\alpha_2^{t-(j-1)}}{(\alpha_2^{t+1}+1)^{j-2}} \quad \cdots \quad \frac{\alpha_{n-1}^{t-(j-1)}}{(\alpha_{n-1}^{t+1}+1)^{j-2}} \quad 0 \right]$$

belong to submatrices  $h_{j-1}$  and  $h_{j-2}$ , respectively. Similar relations can be written for auxiliary  $j-3$  rows with elements

$$\left\{ \frac{\alpha_i^{2t-(j-2)}}{(\alpha_i^{t+1}+1)^{j-2}} \right\}, \left\{ \frac{\alpha_i^{3t-(j-3)}}{(\alpha_i^{t+1}+1)^{j-2}} \right\}, \dots, \left\{ \frac{\alpha_i^{(j-2)t-2}}{(\alpha_i^{t+1}+1)^{j-2}} \right\},$$

$i = 1, \dots, n$

by using the rows in submatrices  $h_{j-1}$  and  $h_{j-2}$  and auxiliary  $j-4$  rows with elements

$$\left\{ \frac{\alpha_i^{2t-(j-2)}}{(\alpha_i^{t+1}+1)^{j-3}} \right\}, \left\{ \frac{\alpha_i^{3t-(j-3)}}{(\alpha_i^{t+1}+1)^{j-3}} \right\}, \dots, \left\{ \frac{\alpha_i^{(j-3)t-3}}{(\alpha_i^{t+1}+1)^{j-3}} \right\},$$

$i = 1, \dots, n.$

Thus, the number of relations decreases by one with every step, so that we obtain for the last step:

$$\frac{\alpha_i^{2t-(j-2)}}{(\alpha_i^{t+1}+1)^2} = \frac{\alpha_i^{t-(j-1)}}{(\alpha_i^{t+1}+1)} - \frac{\alpha_i^{t-(j-1)}}{(\alpha_i^{t+1}+1)^2}$$

for all  $i = 1, \dots, n$ .

Here the rows

$$\left[ \frac{\alpha_1^{t-(j-1)}}{\alpha_1^{t+1}+1} \quad \frac{\alpha_2^{t-(j-1)}}{\alpha_2^{t+1}+1} \quad \cdots \quad \frac{\alpha_{n-1}^{t-(j-1)}}{\alpha_{n-1}^{t+1}+1} \quad 0 \right]$$

and

$$\left[ \frac{\alpha_1^{t-(j-1)}}{(\alpha_1^{t+1}+1)^2} \quad \frac{\alpha_2^{t-(j-1)}}{(\alpha_2^{t+1}+1)^2} \quad \cdots \quad \frac{\alpha_{n-1}^{t-(j-1)}}{(\alpha_{n-1}^{t+1}+1)^2} \quad 0 \right]$$

belong to the submatrices  $h_1$  and  $h_2$ , respectively.

Hence, the number of  $q$ -ary rows equal to  $\delta_j = l + 2lj$  can be deleted from the submatrix  $h_j$  because they are linearly dependent on other rows of this and previous submatrices.

*Remark 3:* Note that we used the row

$$\left[ \frac{\alpha_1^{t-(j-1)}}{\alpha_1^{t+1}+1} \quad \frac{\alpha_2^{t-(j-1)}}{\alpha_2^{t+1}+1} \quad \cdots \quad \frac{\alpha_{n-1}^{t-(j-1)}}{\alpha_{n-1}^{t+1}+1} \quad 0 \right]$$

of the submatrix  $h_1$  when representing the row of the submatrix  $h_j$ ,  $j < q$ .

The total number of  $q$ -ary rows in submatrices  $h_1, h_2, \dots, h_j$ ,  $j < q$  that are linearly dependent and satisfy the above presented relations is equal to  $\Delta_j$ :

$$\Delta_j = \delta_1 + \delta_2 + \dots + \delta_j = lj + 2l(1 + 2 + \dots + j) = lj + 2l \frac{j(j+1)}{2},$$

$$\Delta_j = 2l \frac{j(j+3)}{2}. \quad (16)$$

So, the estimation for the redundancy of the cumulative-separable  $(L, G)$ -code with  $G(x) = G_6^{(j)}(x) = (x^{t+1} + 1)^j$ ,  $1 < j \leq q-2$  and the set  $L_6 = \{GF(t^2) \setminus \{\alpha : G_6(\alpha) = 0\}\}$  is given by  $r_6^{(j)}$ :

$$r_6^{(j)} \leq 2l \left( j(t+1) - \frac{j(j+3)}{2} \right). \quad (17)$$

Consequently, the estimation on the dimension of this code is

$$k_6^{(j)} \geq t^2 - t - 1 - 2lj \left( t + 1 - \frac{j+3}{2} \right). \quad (18)$$

Of special interest is the case  $j = q-1, q$ , i.e., the case when the submatrix  $h_{q-1}$  is presented in the parity check matrix  $H_6$  of the code.

Any row

$$\left[ \begin{array}{cccc} \frac{\alpha_1^{\frac{f}{q}+1}}{\alpha_1^{t+1}+1} & \frac{\alpha_2^{\frac{f}{q}+1}}{\alpha_2^{t+1}+1} & \dots & \frac{\alpha_{n-1}^{\frac{f}{q}+1}}{\alpha_{n-1}^{t+1}+1} & 0 \end{array} \right], j = 1, \dots, q-1$$

from  $h_1$  can be obtained as a linear combination of the rows of submatrices  $h_{q-1}, h_{q-2}, \dots, h_1$ .

In other words, the estimation of the redundancy of this code can be improved by the value  $\theta_{q-1} = (q-1)2l$  in case there is  $h_{q-1}$  in the cumulative-separable parity check matrix.

*Remark 4:* According to Remarks 1,2,3, we did not use the above considered rows of the submatrix  $h_1$  for the analysis of the linear dependence of rows in the parity check matrix  $H_6$  of the cumulative-separable  $\Gamma(L_6, G_6^{(q)})$ -code.

First of all, let us prove the following statement.

*Lemma 8:* All rows of the submatrix  $h_1$

$$\left[ \begin{array}{cccc} \frac{\alpha_1^{\frac{f}{q}+1}}{\alpha_1^{t+1}+1} & \frac{\alpha_2^{\frac{f}{q}+1}}{\alpha_2^{t+1}+1} & \dots & \frac{\alpha_{n-1}^{\frac{f}{q}+1}}{\alpha_{n-1}^{t+1}+1} & 0 \end{array} \right]$$

for  $f = 1, \dots, q-1$  can be obtained as a linear combination of the corresponding row of the submatrix  $h_1$

$$\left[ \begin{array}{cccc} \frac{\alpha_1^{t-f\frac{1}{q}}}{\alpha_1^{t+1}+1} & \frac{\alpha_2^{t-f\frac{1}{q}}}{\alpha_2^{t+1}+1} & \dots & \frac{\alpha_{n-1}^{t-f\frac{1}{q}}}{\alpha_{n-1}^{t+1}+1} & 0 \end{array} \right],$$

and auxiliary rows

$$\left[ \begin{array}{cccc} \frac{\alpha_1^{(f-j)t+q-j}}{(\alpha_1^{t+1}+1)^{q-1}} & \frac{\alpha_2^{(f-j)t+q-j}}{(\alpha_2^{t+1}+1)^{q-1}} & \dots & \frac{\alpha_{n-1}^{(f-j)t+q-j}}{(\alpha_{n-1}^{t+1}+1)^{q-1}} & 0 \end{array} \right],$$

where  $j = 1, \dots, f$ .

*Proof:*

It is easy to observe that the row

$$\left[ \begin{array}{cccc} \frac{\alpha_1^{ft+q}}{(\alpha_1^{t+1}+1)^q} & \frac{\alpha_2^{ft+q}}{(\alpha_2^{t+1}+1)^q} & \dots & \frac{\alpha_{n-1}^{ft+q}}{(\alpha_{n-1}^{t+1}+1)^q} & 0 \end{array} \right]$$

results from the row

$$\left[ \begin{array}{cccc} \frac{\alpha_1^{\frac{f}{q}+1}}{\alpha_1^{t+1}+1} & \frac{\alpha_2^{\frac{f}{q}+1}}{\alpha_2^{t+1}+1} & \dots & \frac{\alpha_{n-1}^{\frac{f}{q}+1}}{\alpha_{n-1}^{t+1}+1} & 0 \end{array} \right].$$

by raising of all its elements to the power  $q$ .

Now let us write a system of recurrent equations:

$$\begin{aligned} \frac{\alpha_i^{ft+q}}{(\alpha_i^{t+1}+1)^q} &= \frac{\alpha_i^{(f-1)t+q-1}}{(\alpha_i^{t+1}+1)^{q-1}} - \frac{\alpha_i^{(f-1)t+q-1}}{(\alpha_i^{t+1}+1)^q}, \\ \frac{\alpha_i^{(f-1)t+q-1}}{(\alpha_i^{t+1}+1)^{q-1}} &= \frac{\alpha_i^{(f-2)t+q-2}}{(\alpha_i^{t+1}+1)^{q-2}} - \frac{\alpha_i^{(f-2)t+q-2}}{(\alpha_i^{t+1}+1)^{q-1}}, \\ \frac{\alpha_i^{(f-2)t+q-2}}{(\alpha_i^{t+1}+1)^{q-2}} &= \frac{\alpha_i^{(f-3)t+q-3}}{(\alpha_i^{t+1}+1)^{q-3}} - \frac{\alpha_i^{(f-3)t+q-3}}{(\alpha_i^{t+1}+1)^{q-2}}, \\ &\vdots \\ \frac{\alpha_i^{(f-r)t+q-r}}{(\alpha_i^{t+1}+1)^{q-r}} &= \frac{\alpha_i^{(f-r-1)t+q-r-1}}{(\alpha_i^{t+1}+1)^{q-r-1}} - \frac{\alpha_i^{(f-r-1)t+q-r-1}}{(\alpha_i^{t+1}+1)^{q-r}}, \\ &\vdots \\ \frac{\alpha_i^{t+q-(f-1)}}{(\alpha_i^{t+1}+1)^q} &= \frac{\alpha_i^{q-f}}{(\alpha_i^{t+1}+1)^{q-1}} - \frac{\alpha_i^{q-f}}{(\alpha_i^{t+1}+1)^q}. \end{aligned}$$

Obviously, the second term in the right part of the last equation can be obtained as the  $qt$ -th power of the element

$$\frac{\alpha_i^{t-f\frac{1}{q}}}{\alpha_i^{t+1}+1}; \quad \left( \frac{\alpha_i^{t-f\frac{1}{q}}}{\alpha_i^{t+1}+1} \right)^{qt} = \frac{\alpha_i^{q-f}}{(\alpha_i^{t+1}+1)^q}, \quad \text{where } t - f\frac{1}{q} < t - 1 \quad \text{for all } f = 1, \dots, q-1.$$

Now write a representation of the above considered auxiliary rows in terms of other auxiliary rows which elements have smaller degree  $j$  of denominator  $(\alpha_i^{t+1}+1)^j$ .

*Lemma 9:* All auxiliary rows with components  $\frac{\alpha_i^{(f-1)t+q-1}}{(\alpha_i^{t+1}+1)^{q-1}}$ ,  $f = 2, \dots, q-1$  can be obtained as a linear combination of corresponding auxiliary rows with the components  $\frac{\alpha_i^{(v-2)t+q-2}}{(\alpha_i^{t+1}+1)^{q-2}}$ ,  $v = 3, \dots, q-1$  and rows from submatrices  $h_{q-1}, h_{q-2}$ :

$$\left[ \begin{array}{cccc} \frac{\alpha_1^{q-f}}{(\alpha_1^{t+1}+1)^{q-1}} & \frac{\alpha_2^{q-f}}{(\alpha_2^{t+1}+1)^{q-1}} & \dots & \frac{\alpha_{n-1}^{q-f}}{(\alpha_{n-1}^{t+1}+1)^{q-1}} & 0 \end{array} \right],$$

$$\left[ \begin{array}{cccc} \frac{\alpha_1^{q-f}}{(\alpha_1^{t+1}+1)^{q-2}} & \frac{\alpha_2^{q-f}}{(\alpha_2^{t+1}+1)^{q-2}} & \dots & \frac{\alpha_{n-1}^{q-f}}{(\alpha_{n-1}^{t+1}+1)^{q-2}} & 0 \end{array} \right].$$

*Proof:* Similarly to the previous Lemma, we can write a system of recurrent equations:

$$\begin{aligned} \frac{\alpha_i^{(f-1)t+q-1}}{(\alpha_i^{t+1}+1)^{q-1}} &= \frac{\alpha_i^{(f-2)t+q-2}}{(\alpha_i^{t+1}+1)^{q-2}} - \frac{\alpha_i^{(f-2)t+q-2}}{(\alpha_i^{t+1}+1)^{q-1}}, \\ \frac{\alpha_i^{(f-2)t+q-2}}{(\alpha_i^{t+1}+1)^{q-2}} &= \frac{\alpha_i^{(f-3)t+q-3}}{(\alpha_i^{t+1}+1)^{q-3}} - \frac{\alpha_i^{(f-3)t+q-3}}{(\alpha_i^{t+1}+1)^{q-2}}, \\ \frac{\alpha_i^{(f-3)t+q-3}}{(\alpha_i^{t+1}+1)^{q-3}} &= \frac{\alpha_i^{(f-4)t+q-4}}{(\alpha_i^{t+1}+1)^{q-4}} - \frac{\alpha_i^{(f-4)t+q-4}}{(\alpha_i^{t+1}+1)^{q-3}}, \\ &\vdots \\ \frac{\alpha_i^{(f-r)t+q-r}}{(\alpha_i^{t+1}+1)^{q-r}} &= \frac{\alpha_i^{(f-r-1)t+q-r-1}}{(\alpha_i^{t+1}+1)^{q-r-1}} - \frac{\alpha_i^{(f-r-1)t+q-r-1}}{(\alpha_i^{t+1}+1)^{q-r}}, \\ &\vdots \\ \frac{\alpha_i^{t+q-(f-1)}}{(\alpha_i^{t+1}+1)^{q-1}} &= \frac{\alpha_i^{q-f}}{(\alpha_i^{t+1}+1)^{q-2}} - \frac{\alpha_i^{q-f}}{(\alpha_i^{t+1}+1)^{q-1}}. \end{aligned}$$

It is clear that the elements in the right part of the last equation are components of the corresponding rows of submatrices  $h_{q-1}$  and  $h_{q-2}$ . ■

*Remark 5:* Note that in order to represent  $q-2$  auxiliary rows with components in the form  $\frac{\alpha_i^{(f-r)t+q-r}}{(\alpha_i^{t+1}+1)^{q-1}}$  we used  $q-3$  new auxiliary rows with components in the form  $\frac{\alpha_i^{(f-r-1)t+q-r-1}}{(\alpha_i^{t+1}+1)^{q-2}}$ .

Recurrent equations for auxiliary rows with a reduced denominator power of the components and the corresponding rows of submatrices  $h_{q-1}, h_{q-2}, \dots, h_{q-f}$  can be extended up to the last step for which we have the following relation:

$$\frac{\alpha_i^{t+q-(f-1)}}{(\alpha_i^{t+1}+1)^{q-(f-1)}} = \frac{\alpha_i^{q-f}}{(\alpha_i^{t+1}+1)^{q-f}} - \frac{\alpha_i^{q-f}}{(\alpha_i^{t+1}+1)^{q-(f-1)}},$$

where the rows with components  $\frac{\alpha_i^{q-f}}{(\alpha_i^{t+1}+1)^{q-f}}$ ,  $\frac{\alpha_i^{q-f}}{(\alpha_i^{t+1}+1)^{q-(f-1)}}$  belong to the submatrices  $h_{q-f}, h_{q-(f-1)}$ . Thus, with taking the above statements into account, we can prove the following theorem.

*Theorem 2:* All rows of the submatrix  $h_1$

$$\left[ \begin{array}{cccc} \frac{\alpha_1^{\frac{f}{q}+1}}{\alpha_1^{t+1}+1} & \frac{\alpha_2^{\frac{f}{q}+1}}{\alpha_2^{t+1}+1} & \dots & \frac{\alpha_{n-1}^{\frac{f}{q}+1}}{\alpha_{n-1}^{t+1}+1} & 0 \end{array} \right],$$

■ where  $f = 1, \dots, q-1$ , can be obtained as a linear combina-

tions of the following rows of submatrices  $h_1, h_2, \dots, h_{q-1}$ :

$$\begin{bmatrix} \frac{\alpha_1^{t-f\frac{1}{q}}}{\alpha_1^{t+1}+1} & \frac{\alpha_2^{t-f\frac{1}{q}}}{\alpha_2^{t+1}+1} & \dots & \frac{\alpha_{n-1}^{t-f\frac{1}{q}}}{\alpha_{n-1}^{t+1}+1} & 0 \\ \frac{\alpha_1^{q-f}}{(\alpha_1^{t+1}+1)^2} & \frac{\alpha_2^{q-f}}{(\alpha_2^{t+1}+1)^2} & \dots & \frac{\alpha_{n-1}^{q-f}}{(\alpha_{n-1}^{t+1}+1)^2} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\alpha_1^{q-f}}{(\alpha_1^{t+1}+1)^{q-1}} & \frac{\alpha_2^{q-f}}{(\alpha_2^{t+1}+1)^{q-1}} & \dots & \frac{\alpha_{n-1}^{q-f}}{(\alpha_{n-1}^{t+1}+1)^{q-1}} & 0 \end{bmatrix}.$$

It means that the number of  $q$ -ary rows equal to  $\theta_{q-1} = (q-1)2l$  can be additionally deleted from the submatrix  $h_1$ . Thus, we get the following estimation for the dimension of the cumulative-separable  $(L, G)$ -code with  $G_6^{(q)}(x) = (x^{t+1} + 1)^q$ :

$$k_6^{(q)} \geq n - 2l \left( (q-1)(t+1) - \frac{(q-1)(q+3)}{2} \right)$$

or

$$k_6^{(q)} \geq t^2 - t - 1 - 2l(q-1) \left( t - \frac{3}{2} - \frac{q-2}{2} \right). \quad (19)$$

It should be noted that the previously constructed boundaries for the dimensions of binary and  $q$ -ary separable codes of this subclass are obtained from the above boundary as a special case. By using the technique of embedded code constructing described in [6] for the ternary cumulative-separable  $\Gamma_6(L_6, G_6^{(3)})$ -code with  $L = \{GF(3^4)\{\alpha : G_6(\alpha) = 0\}\}$  and the Goppa polynomial  $G_6^{(3)}(x) = (x^{10} - 1)^3$  we can obtain the following collection of codes (Table III):

$$\Gamma_6(L_6, G_6^{(3)i}), G_6^{(3)i}(x) = (x^{10} - 1)^3(x-1)^i, i \geq 1.$$

Moreover, the best codes from the known ternary block linear ones [3] are present in this collection.

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TABLE II  
PARAMETERS OF THE CUMULATIVE-SEPARABLE CODES  
 $\Gamma_6^{(q)} = (x^{t+1} + 1)^q, t = q^l$

$q$	$l$	$n$	$k$ estimation (19)	$k$ real value	$d \geq q(t+1) + 1$ estimation
3	2	71	15	16	31
3	3	701	401	401	85
3	4	6479	5215	5215	247
5	2	599	247	256	131
5	3	15499	12571	12571	631
7	2	2351	1271	1296	351
11	2	14519	9919	10000	1343

TABLE IV  
PARAMETERS OF THE CUMULATIVE-SEPARABLE CODES  
 $\Gamma_1^{(q)} = (x^{t-1} + 1)^q, t = q^l$

$q$	$l$	$n$	$k$ estimation (Section IV-B)	$k$ real value	$d \geq q(t-1) + 1$ estimation
3	2	73	16	17	25
3	3	703	402	402	79
3	4	6481	5216	5216	241
5	2	601	248	257	121
5	3	15501	12572	12572	621
7	2	2353	1272	1297	337
11	2	14522	9921	10002	1331

TABLE V  
DEPENDENCE OF CUMULATIVE-SEPARABLE CODES DIMENSIONS ON THE CUMULATIVITY ORDERS  $1 \leq i \leq q-1$  FOR  $q = 3, l = 2$  AND  $t = 9$

$i$	$\Gamma_1^{(i)}$ $n_1 = 73$	$\Gamma_1^{*(i)}$ $n_1^* = 72$	$\Gamma_2^{(i)} \equiv \Gamma_3^{(i)}$ $n_2 = 72$	$C_3^{*(i)} \equiv \Gamma_4^{*(i)}$ $n_3^* = 71$	$\Gamma_5^{(i)} \equiv \Gamma_6^{(i)}$ $n_5 = 71$
1	42	41	39	39	37
2	17	16	16	16	16

TABLE VI  
DEPENDENCE OF CUMULATIVE-SEPARABLE CODES DIMENSIONS ON THE CUMULATIVITY ORDERS  $1 \leq i \leq q-1$  FOR  $q = 3, l = 3$  AND  $t = 27$

$i$	$\Gamma_1^{(i)}$ $n_1 = 703$	$\Gamma_1^{*(i)}$ $n_1^* = 702$	$\Gamma_2^{(i)} \equiv \Gamma_3^{(i)}$ $n_2 = 702$	$C_3^{*(i)} \equiv \Gamma_4^{*(i)}$ $n_3^* = 701$	$\Gamma_5^{(i)} \equiv \Gamma_6^{(i)}$ $n_5 = 701$
1	549	548	545	545	542
2	402	401	401	401	401

TABLE VII  
DEPENDENCE OF CUMULATIVE-SEPARABLE CODES DIMENSIONS ON THE CUMULATIVITY ORDERS  $1 \leq i \leq q-1$  FOR  $q = 5, l = 2$  AND  $t = 25$

$i$	$\Gamma_1^{(i)}$ $n_1 = 601$	$\Gamma_1^{*(i)}$ $n_1^* = 600$	$\Gamma_2^{(i)} \equiv \Gamma_3^{(i)}$ $n_2 = 600$	$C_3^{*(i)} \equiv \Gamma_4^{*(i)}$ $n_3^* = 599$	$\Gamma_5^{(i)} \equiv \Gamma_6^{(i)}$ $n_5 = 599$
1	506	505	503	503	501
2	412	411	409	409	407
3	322	321	319	319	317
4	257	256	256	256	256

TABLE VIII  
DEPENDENCE OF CUMULATIVE-SEPARABLE CODES DIMENSIONS ON THE CUMULATIVITY ORDERS  $1 \leq i \leq q-1$  FOR  $q = 7, l = 2$  AND  $t = 49$

$i$	$\Gamma_1^{(i)}$ $n_1 = 2353$	$\Gamma_1^{*(i)}$ $n_1^* = 2352$	$\Gamma_2^{(i)} \equiv \Gamma_3^{(i)}$ $n_2 = 2352$	$C_3^{*(i)} \equiv \Gamma_4^{*(i)}$ $n_3^* = 2351$	$\Gamma_5^{(i)} \equiv \Gamma_6^{(i)}$ $n_5 = 2351$
1	2162	2161	2159	2159	2157
2	1972	1971	1969	1969	1967
3	1786	1785	1783	1783	1781
4	1604	1603	1601	1601	1599
5	1426	1425	1423	1423	1421
6	1297	1296	1296	1296	1296

TABLE I  
CUMULATIVE-SEPARABLE CODES

$i$	$\Gamma_1^{(i)}$ $n_1 = n_1^* + 1$	$\Gamma_1^{*(i)}$ $n_1^* = n_2$	$\Gamma_2^{(i)} \equiv \Gamma_3^{(i)}$ $n_2 = n_3 = n_3^* + 1$	$C_3^{*(i)} \equiv \Gamma_4^{*(i)}$ $n_3^* = n_4^* = n_5$	$\Gamma_5^{(i)} \equiv \Gamma_6^{(i)}$ $n_5 = n_6 = t^2 - t - 1$
$\leq q - 2$	$k_1^{(i)} = k_1^{*(i)} + 1,$ $d_1^{(i)} = i(t - 1) + 1$	$k_1^{*(i)} \leq k_2^{(i)} + l,$ $d_1^{(i)} \leq d_1^{*(i)} \leq d_2^{(i)}$	$k_2^{(i)} = k_3^{(i)} = k_3^{*(i)},$ $d_2^{(i)} = d_3^{(i)} \geq d_3^{*(i)}$	$k_3^{*(i)} = k_4^{*(i)} \leq k_5^{(i)} + l,$ $d_3^{*(i)} = d_4^{*(i)} \leq d_5^{(i)}$	$k_5^{(i)} = k_6^{(i)},$ $k_6^{(i)} \geq t^2 - t - 1 -$ $2lj \left( t + 1 - \frac{j+3}{2} \right),$ $d_5^{(i)} = d_6^{(i)} = i(t + 1) + 1$
$q, q - 1$	$k_1^{(q)} = k_1^{*(q)} + 1,$ $d_1^{(q)} \geq q(t - 1) + 1$	$k_1^{*(q)} = k_2^{(q)},$ $d_1^{(q)} \leq d_1^{*(q)} = d_2^{(q)}$	$k_2^{(q)} = k_3^{(q)} = k_3^{*(q)},$ $d_2^{(q)} = d_3^{(q)} \geq d_3^{*(q)}$	$k_3^{*(q)} = k_4^{*(q)} = k_5^{(q)},$ $d_3^{*(q)} = d_4^{*(q)} = d_5^{(q)}$	$k_5^{(q)} = k_6^{(q)},$ $k_6^{(q)} \geq t^2 - t - 1 -$ $2l(q - 1) \left( t - \frac{3}{2} - \frac{q-2}{2} \right),$ $d_5^{(q)} = d_6^{(q)} \geq q(t + 1) + 1$

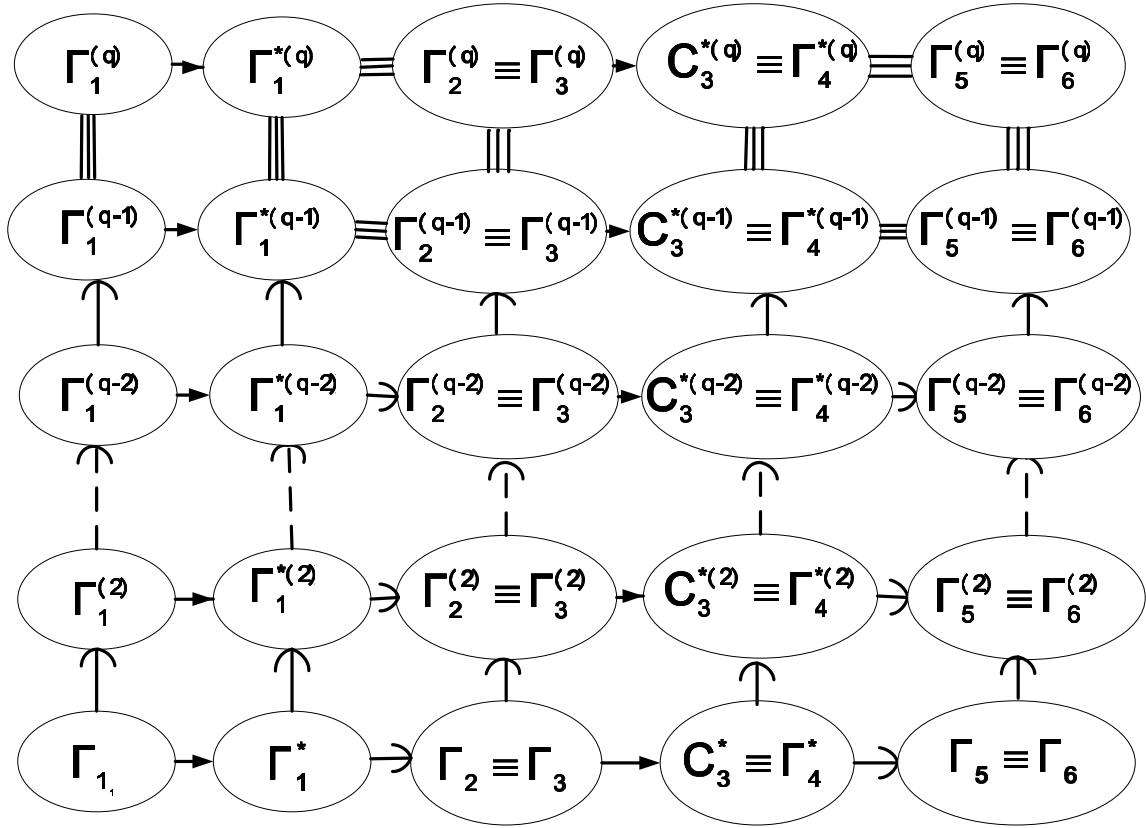


Fig. 1. Embedded and equivalent cumulative-separable codes

TABLE III  
SEQUENCE OF EMBEDDED TERNARY CODES DERIVED FROM THE CODE  $\Gamma_6(L_6, G_6^{(3)})$ ,  $G_6^{(3)} = (x^{10} - 1)^3$

$i$	$G_6^{(3)i} = G_6^{(3)}(x - 1)^i$	$n$	$k$	$d$	$d$ [3]
0	$(x^{10} - 1)^3$	71	16	31	31
1	$(x^{10} - 1)^3(x - 1)$	71	15	33	32
4	$(x^{10} - 1)^3(x - 1)^4$	71	11	35	36
10	$(x^{10} - 1)^3(x - 1)^{10}$	71	7	42	42
13	$(x^{10} - 1)^3(x - 1)^{13}$	71	5	44	45